

From 3-moves to Lagrangian tangles and cubic skein modules

Józef H. Przytycki

Abstract

We present an expanded version of four talks describing recent developments in Knot Theory to which the author contributed¹. We discuss several open problems in classical Knot Theory and we develop techniques that allow us to study them: Lagrangian tangles, skein modules and Burnside groups. The method of Burnside groups of links was discovered and developed only half a year after the last talk was delivered in Kananaskis².

1 Open problems in Knot Theory that everyone can try to solve

When did Knot Theory start? Was it in 1794 when C. F. Gauss³ (1777-1855) copied figures of knots from a book written in English (Fig.1.1)?

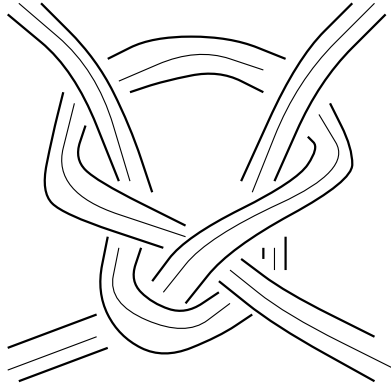


Fig. 1.1; Gauss' meshing knot from 1794

¹Containing several results that are not yet published elsewhere.

²The first three talks were delivered at International Workshop on Graphs – Operads – Logic; Cuautitlán, Mexico, March 12-16, 2001 and the fourth talk “Symplectic Structures on Colored Tangles” at the workshop New Techniques in Topological Quantum Field Theory; Calgary/Kananaskis, August 22-27, 2001.

³Gauss' notebooks contain several drawings related to knots, for example a braid with complex coordinate description (see [Ep-1, P-14]) or the mysterious “framed tangle” which is published here for the first time, see Fig.1.2. [Ga].

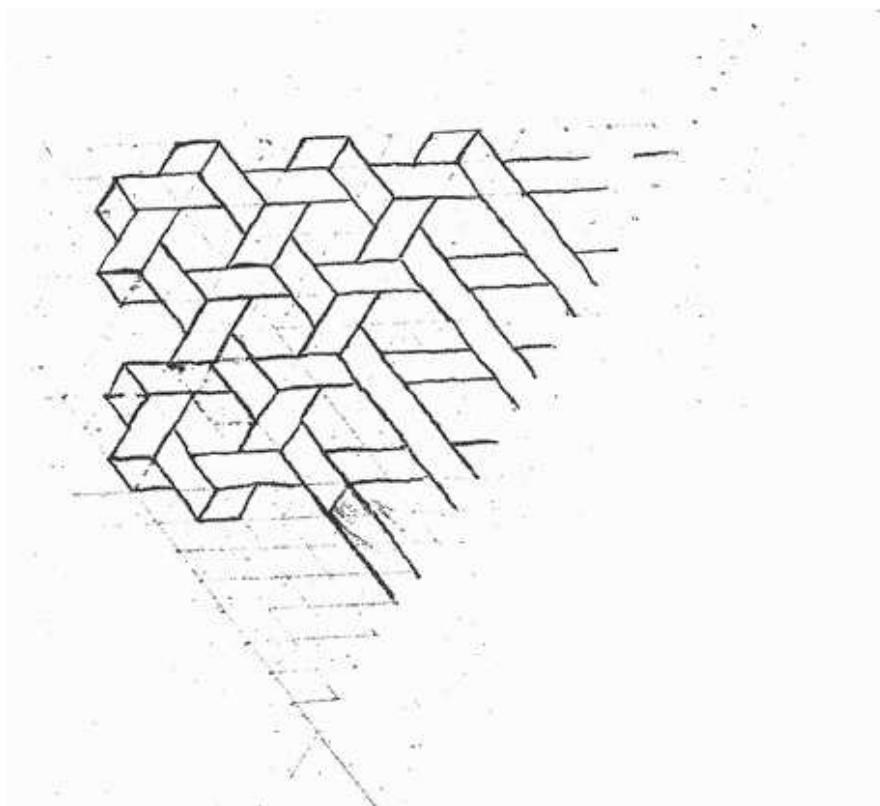


Fig. 1.2; Framed tangle from Gauss'notebook [Ga]

Or was it before that, in 1771, when A-T. Vandermonde (1735-1796) considered knots and braids as a part of Leibniz's *analysis situs*?

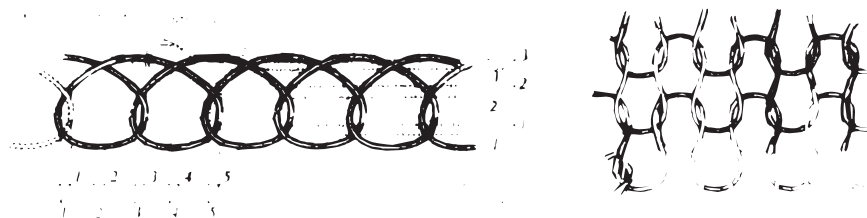


Fig. 1.3; Vandermonde drawings of 1771

Perhaps engravings by Leonardo da Vinci⁴ (1452-1519) [Mac] and woodcuts by Albrecht Dürer⁵ (1471-1528) [Dur-1, Ha] should also be taken into account, Fig.1.4.

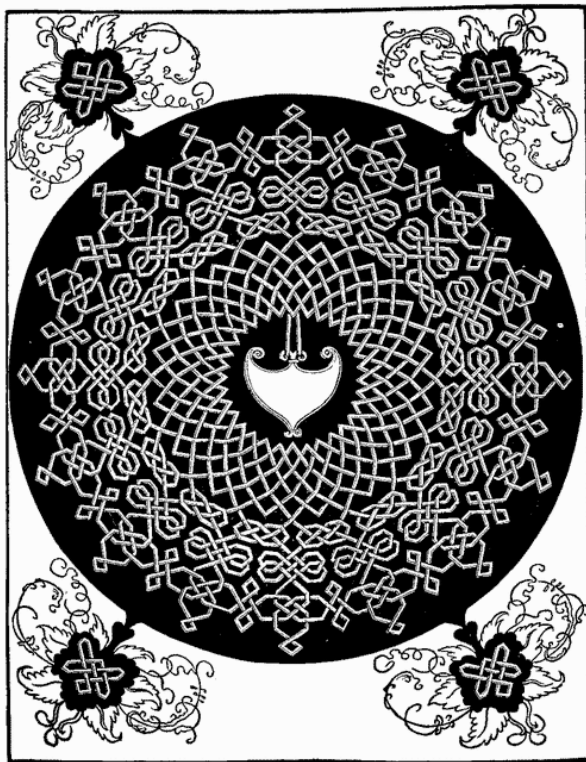


Fig. 1.4; A knot by Dürer [Ku]; c. 1505-1507

One can go back in time even further to ancient Greece where surgeons

⁴Giorgio Vasari writes in [Va]: “[Leonardo da Vinci] spent much time in making a regular design of a series of knots so that the cord may be traced from one end to the other, the whole filling a round space. There is a fine engraving of this most difficult design, and in the middle are the words: Leonardus Vinci Academia.”

⁵“Another great artist with whose works Dürer now became acquainted was Leonardo da Vinci. It does not seem likely that the two artists ever met, but he may have been brought into relation with him through Luca Pacioli, the author of the book *De Divina Proportione*, which appeared at Venice in 1509, and an intimate friend of the great Leonardo. Dürer would naturally be deeply interested in the proportion theories of Leonardo and Pacioli. He was certainly acquainted with some engravings of Leonardo’s school, representing a curious circle of concentric scrollwork on a black ground, one of them entitled *Accademia Leonardi Vinci*; for he himself executed six woodcuts in imitation, the *Six Knots*, as he calls them himself. Dürer was amused by and interested in all scientific or mathematical problems...” From: <http://www.cwru.edu/edocs/7/258.pdf>, compare [Dur-2].

considered sling knots, Fig.1.5 [Da, P-14].

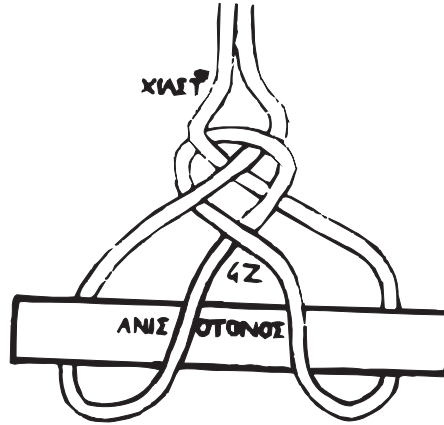


Fig. 1.5; A sling knot of Heraclas

Moreover, we can appreciate ancient stamps and seals with knots and links as their motifs. The oldest examples that I am aware of are from the pre-Hellenic Greece. Excavations at Lerna by the American School of Classical Studies under the direction of Professor J. L. Caskey (1952-1958) discovered two rich deposits of clay seal-impressions. The second deposit dated from about 2200 BC contains several impressions of knots and links⁶ [Hig, Hea, Wie] (see Fig.1.6).

⁶The early Bronze Age in Greece is divided, as in Crete and the Cyclades, into three phases. The second phase lasted from 2500 to 2200 BC, and was marked by a considerable increase in prosperity. There were palaces at Lerna, and Tiryns, and probably elsewhere, in contact with the Second City of Troy. The end of this phase (in the Peloponnese) was brought about by invasion and mass burnings. The invaders are thought to be the first speakers of the Greek language to arrive in Greece.



Fig. 1.6; A seal-impression from the House of the Tiles in Lerna [Hig].

As we see Knot Theory has a long history but despite this, or maybe because of this, one still can find inspiring elementary open problems. These problems are not just interesting puzzles but they lead to an interesting theory.

In this section our presentation is absolutely elementary. Links are circles embedded in our space, R^3 , up to topological deformation, that is, two links are equivalent if one can be deformed into the other in space without cutting and pasting. We represent links using their plane diagrams.

First we introduce the concept of an n move on a link.

Definition 1.1 *An n -move on a link is a local transformation of the link illustrated in Figure 1.7.*

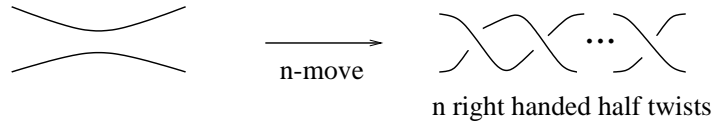
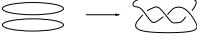


Fig. 1.7; n -move

In our convention, the part of the link outside of the disk in which the move takes place, remains unchanged. One should stress that an n -move can change the topology of the link. For example  illustrates a 3-move.

Definition 1.2 *We say that two links, L_1 and L_2 , are n -move equivalent if one can obtain L_2 from L_1 by a finite number of n -moves and $(-n)$ -moves (inverses of n -moves).*

If we work with diagrams of links then the topological deformation of links is captured by Reidemeister moves, that is, two diagrams represent the same link in space if and only if one can obtain one of them from the other by a sequence of Reidemeister moves:

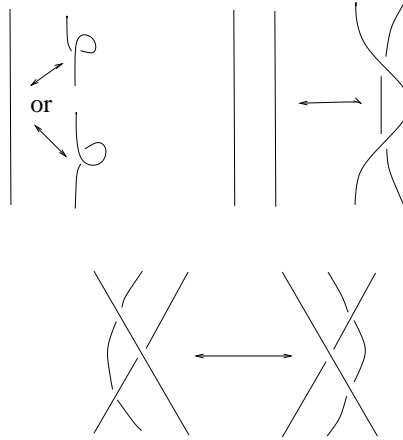
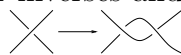


Fig. 1.8; Reidemeister moves

Thus, we say that two diagrams, D_1 and D_2 , are n -move equivalent if one can be obtained from the other by a sequence of n -moves, their inverses and Reidemeister moves. To illustrate this, we show that the move 

is the result of an application of a 3-move followed by the second Reidemeister move (Fig.1.9).

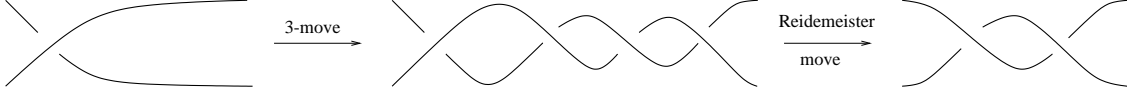


Fig. 1.9

Conjecture 1.3 (Montesinos-Nakanishi)

Every link is 3-move equivalent to a trivial link.

Yasutaka Nakanishi proposed this conjecture in 1981. José Montesinos analyzed 3-moves before, in connection with 3-fold dihedral branch coverings, and asked a related but different question⁷ [Mo-2].

Examples 1.4 (i) *Trefoil knots (left- and right-handed) are 3-move equivalent to the trivial link of two components:*

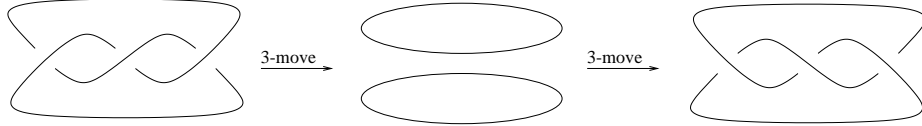


Fig. 1.10

(ii) *The figure eight knot (4_1) and the knot 5_2 are 3-move equivalent to the trivial knot:*

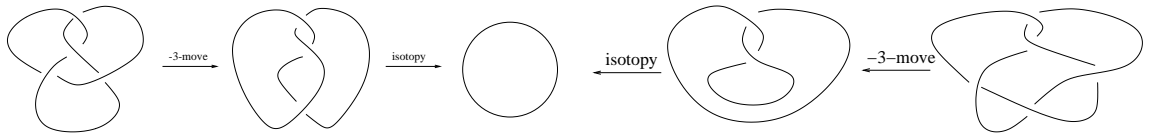


Fig. 1.11

(iii) *The knot 5_1 and the Hopf link are 3-move equivalent to the trivial knot⁸:*

⁷“Is there a set of moves which do not change the covering manifold and such that if two colored links have the same covering they are related by a finite sequence of those moves?”

⁸One can show that the knot 5_1 cannot be reduced to the trivial knot by one ± 3 -move. To see this one can use the Goeritz matrix approach to the classical signature ($|\sigma(5_1)| = 4$), see [Go, G-L].

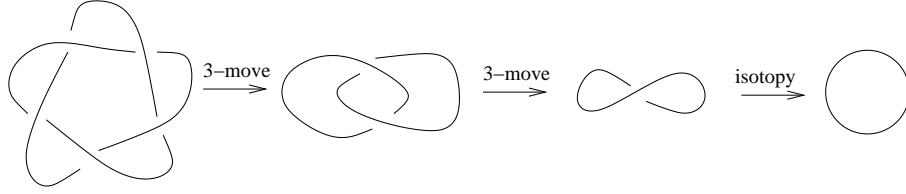


Fig. 1.12

We will show later, in this section, that different trivial links are not 3-move equivalent. However, in order to achieve this conclusion we need an invariant of links preserved by 3-moves and differentiating trivial links. Fox 3-coloring is such an invariant. We will introduce it later (today in its simplest form and, in the second lecture, in a more general context of Fox n -colorings and Alexander-Burau-Fox colorings).

Now let us present some other related conjectures.

Conjecture 1.5

Any 2-tangle is 3-move equivalent to one of the four 2-tangles shown in Fig.1.13 with possible additional trivial components.

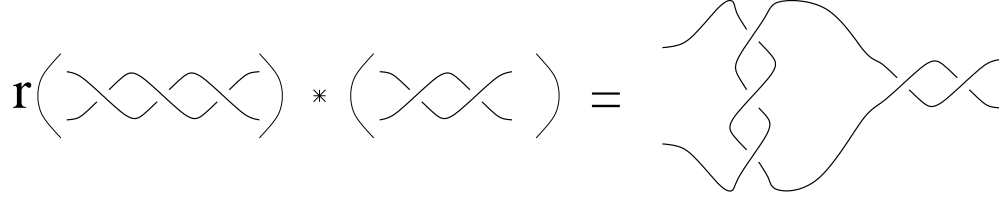


Fig. 1.13

The Montesinos-Nakanishi conjecture follows from Conjecture 1.5. More generally if Conjecture 1.5 holds for some class of 2-tangles, then Conjecture 1.3 holds for any link obtained by closing elements of this class, without introducing any new crossing. The simplest interesting class of tangles for which Conjecture 1.5 holds are algebraic tangles in the sense of Conway (I call them 2-algebraic tangles and in the next section present a generalization). Conjecture 1.5 can be proved by induction for 2-algebraic tangles. I will leave the proof to you as a pleasant exercise (compare Proposition 1.9). The definition you need is as follows

Definition 1.6 ([Co, B-S]) *The family of 2-algebraic tangles is the smallest family of 2-tangles satisfying*

- (i) Any 2-tangle with 0 or 1 crossing is 2-algebraic.
(ii) If A and B are 2-algebraic tangles then the 2-tangle $r^i(A) * r^j(B)$ is also 2-algebraic, where r denotes the counterclockwise rotation of a tangle by 90° along the z -axis, and $*$ denotes the horizontal composition of tangles (see the figure below).



A link is called 2-algebraic if it can be obtained from a 2-algebraic tangle by closing its ends without introducing crossings⁹.

The Montesinos-Nakanishi 3-move conjecture has been proved by my students Qi Chen and Tatsuya Tsukamoto for many special families of links [Che, Tsu, P-Ts]. In particular, Chen proved that the conjecture holds for all 5-braid links except possibly one family, containing the square of the center of the 5-braid group, $\Delta_5^4 = (\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$. He also found a reduction by ± 3 -moves of Δ_5^4 to the 5-braid link, $(\sigma_1^{-1}\sigma_2\sigma_3\sigma_4^{-1}\sigma_3)^4$, with 20 crossings¹⁰, Fig.1.14. It is now the smallest known possible counterexample to the Montesinos-Nakanishi 3-move conjecture¹¹.

⁹By joining the top ends and then bottom ends of a tangle T one obtains the link $N(T)$, the *numerator* of T , Fig.1.22, 1.23. Joining the left-hand ends and then right-hand ends produces the *denominator* closure $D(T)$.

¹⁰In the group $B_5/(\sigma_i^3)$ the calculation is as follows: $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10} = (\sigma_1\sigma_2\sigma_3\sigma_4^2\sigma_3\sigma_2\sigma_1)^2(\sigma_2\sigma_3\sigma_4^2\sigma_3\sigma_2)^2(\sigma_3\sigma_4^2\sigma_3)^2 \stackrel{3}{=} (\sigma_1\sigma_2\sigma_3\sigma_4^2\sigma_3\sigma_2\sigma_1)^2(\sigma_2\sigma_3\sigma_4^2\sigma_3\sigma_2)^2 \stackrel{3}{=} (\sigma_1\sigma_2\sigma_3\sigma_4^2\sigma_3\sigma_2\sigma_1)^2(\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3)^2 \stackrel{3}{=} (\sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_3\sigma_1^{-1}\sigma_2\sigma_3\sigma_4^{-1}\sigma_3\sigma_1)^2 \stackrel{3}{=} (\sigma_1^{-1}\sigma_2\sigma_3\sigma_4^{-1}\sigma_3)^4$.

¹¹We proved in [D-P-1] that Chen's link is in fact the counterexample to the Montesinos-Nakanishi 3-move conjecture; see Section 4. We think that it is the smallest such counterexample. We also demonstrated that the 2-parallel of the Borromean rings is not 3-move equivalent to a trivial link. It is still possible that Chen's link with an additional trivial component is 3-move equivalent to the 2-parallel of the Borromean rings.

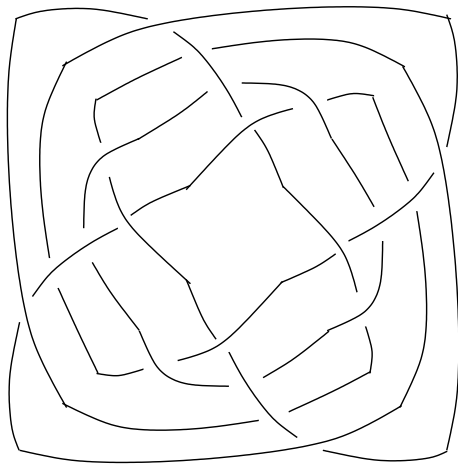


Fig. 1.14

Previously Nakanishi suggested in 1994 (see [Kir]), the 2-parallel of the Borromean rings (a 6-braid with 24 crossings) as a possible counterexample (Fig.1.15).

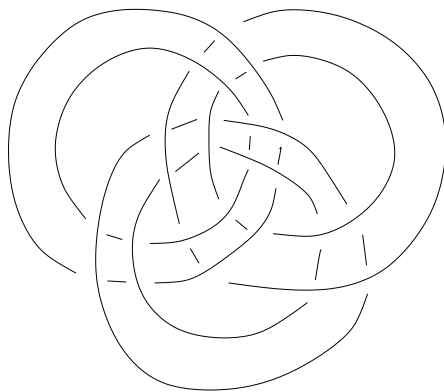


Fig. 1.15

We will return to the discussion of theories motivated by 3-moves tomorrow. Now we will state some conjectures that employs other elementary moves.

Conjecture 1.7 (Nakanishi, 1979)

Every knot is 4-move equivalent to the trivial knot.

Examples 1.8 *Reduction of the trefoil and the figure eight knots is illustrated in Fig.1.16.*

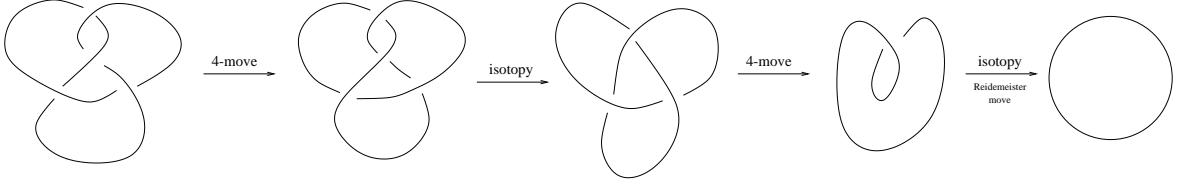


Fig. 1.16

Proposition 1.9 ([P-12]) (i) *Every 2-algebraic tangle without a closed component can be reduced by ± 4 -moves to one of the six basic 2-tangles shown in Fig.1.17.*

(ii) *Every 2-algebraic knot can be reduced by ± 4 -moves to the trivial knot.*

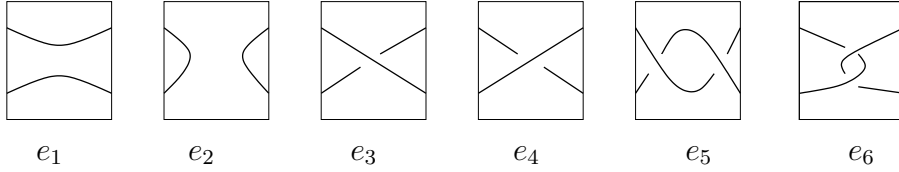


Fig. 1.17

Proof: To prove (i) it suffices to show that every composition (with possible rotation) of tangles presented in Fig.1.17 can be reduced by ± 4 -moves back to one of the tangles in Fig.1.17 or it has a closed component. These can be easily verified by inspection. Fig.1.18 is the multiplication table for basic tangles. We have chosen our basic tangles to be invariant under the rotation r , so it suffices to be able to reduce every tangle of the table to a basic tangle. One example of such a reduction is shown in Fig.1.19. Part (ii) follows from (i). \square

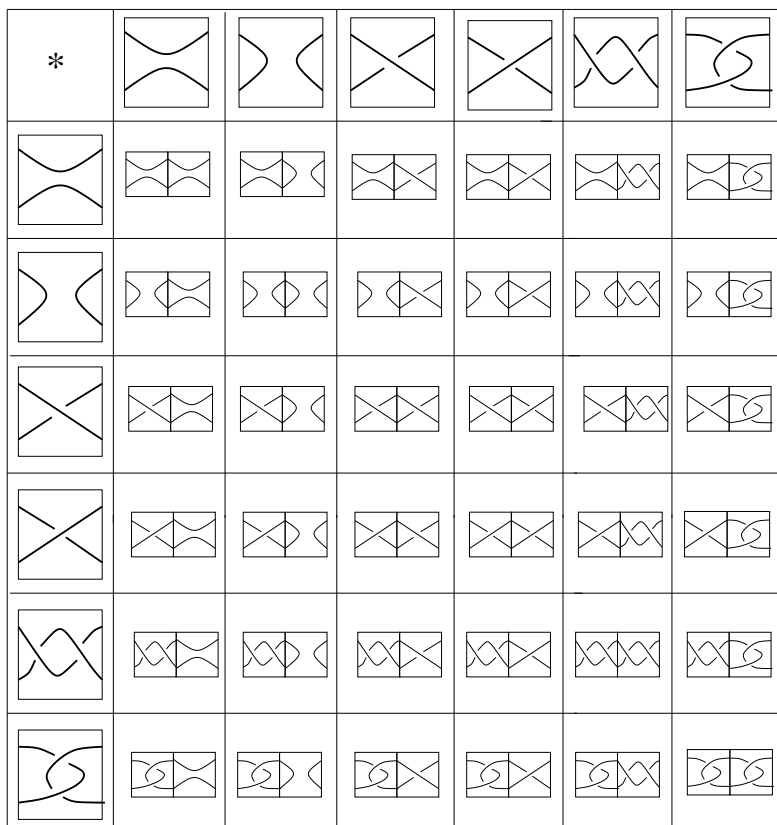


Fig. 1.18

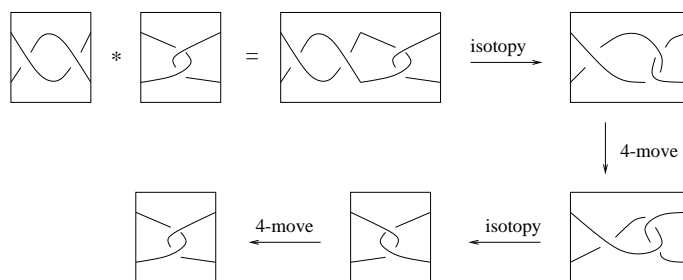


Fig. 1.19

In 1994, Nakanishi began to suspect that a 2-cable of the trefoil knot cannot be simplified by 4-moves [Kir]. However, Nikolaos Askitas was able to simplify this knot [Ask]. Askitas, in turn, suspects that the $(2, 1)$ -cable of the figure eight knot (with 17 crossings) to be the simplest counterexample

to the Nakanishi 4-move conjecture.

Not every link can be reduced to a trivial link by 4-moves. In particular, the linking matrix modulo 2 is preserved by 4-moves. Furthermore, Nakanishi and Suzuki demonstrated that the Borromean rings cannot be reduced to the trivial link of three components [Na-Su].

In 1985, after the seminar talk given by Nakanishi in Osaka, there was discussion about possible generalization of the Nakanishi 4-move conjecture for links. Akio Kawauchi formulated the following question for links

Problem 1.10 ([Kir])

- (i) *Is it true that if two links are link-homotopic¹² then they are 4-move equivalent?*
- (ii) *In particular, is it true that every two component link is 4-move equivalent to the trivial link of two components or to the Hopf link?*

We can extend the argument used in Proposition 1.9 to show:

Theorem 1.11 *Any two component 2-algebraic link is 4-move equivalent to the trivial link of two components or to the Hopf link.*

Proof: Let L be a 2-algebraic link of two components. Therefore, L is built inductively as in Definition 1.6. Consider the first tangle, T , in the construction, which has a closed component (if it happens). The complement T' of T in the link L is also a 2-algebraic tangle but without a closed component. Therefore it can be reduced to one of the 6 basic tangles shown in Fig.1.17, say e_i . Consider the product $T * e_i$. The only nontrivial tangle T to be considered is $e_6 * e_6$ (the last tangle in Fig.1.18). The compositions $(e_6 * e_6) * e_i$ are illustrated in Fig.1.20. The closure of each of these product tangles (the numerator or the denominator) has two components because it is 4-move equivalent to L . We can easily check that it reduces to the trivial link of two components. \square

¹²Two links L_1 and L_2 are *link-homotopic* if one can obtain L_2 from L_1 by a finite number of crossing changes involving only self-crossings of the components.

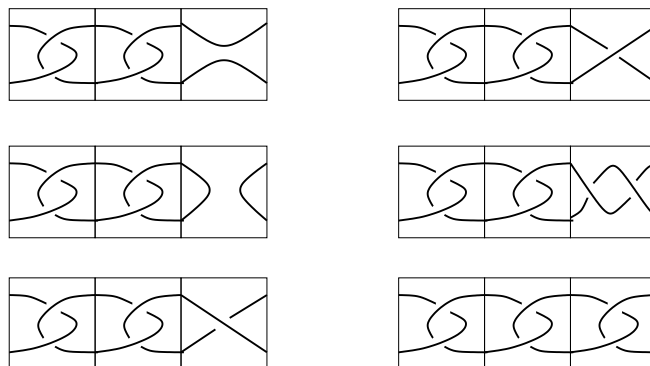


Fig. 1.20

Problem 1.12 (i) Find a (reasonably small) family of 2-tangles with one closed component so that every 2-tangle with one closed component is 4-move equivalent to one of its elements.

(ii) Solve the above problem for 2-algebraic tangles with one closed component.

Nakanishi [Nak-2] pointed out that the “half” 2-cabling of the Whitehead link, \mathcal{W} , Fig.1.21, was the simplest link which he could not reduce to a trivial link by ± 4 -moves but which was link-homotopic to a trivial link¹³.

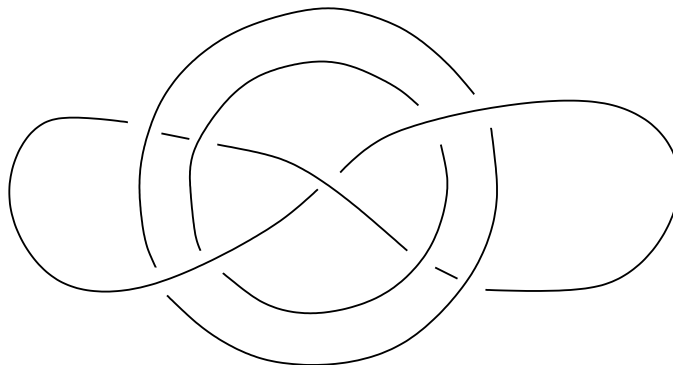


Fig. 1.21

It is shown in Fig.1.22 that the link \mathcal{W} is 2-algebraic. Similarly, the Borromean rings, BR , are 2-algebraic, Fig.1.23 (compare Fig.1.30).

¹³In fact, in June of 2002 we showed that this example cannot be reduced by ± 4 -moves [D-P-2].

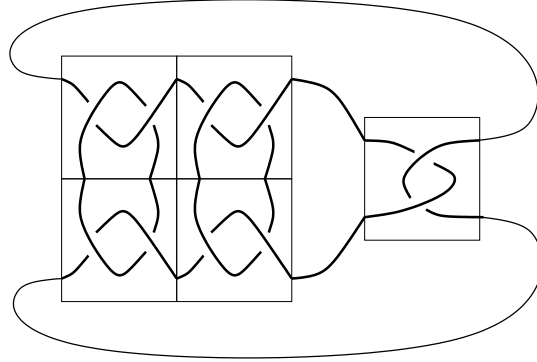


Fig. 1.22; $\mathcal{W} = N(r(r(e_3 * e_3) * r(e_4 * e_4)) * r(r(e_3 * e_3) * r(e_4 * e_4)) * r(e_3 * e_3))$

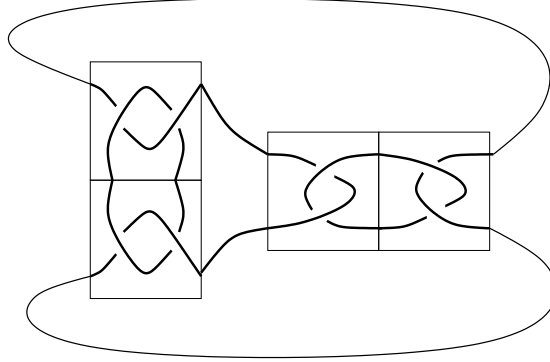


Fig. 1.23; $BR = N(r(r(e_3 * e_3) * r(e_4 * e_4)) * r(e_3 * e_3) * r(e_4 * e_4))$

Problem 1.13 *Is the link \mathcal{W} the only 2-algebraic link of three components (up to 4-move equivalence) which is homotopically trivial but which is not 4-move equivalent to the trivial link of three components?*

We can also prove that the answer to Kawauchi's question is affirmative for closed 3-braids.

- Theorem 1.14** (i) *Every knot which is a closed 3-braid is 4-move equivalent to the trivial knot.*
- (ii) *Every link of two components which is a closed 3-braid is 4-move equivalent to the trivial link of two components or to the Hopf link.*
- (iii) *Every link of three components which is a closed 3-braid is 4-move equivalent either to the trivial link of three components, or to the Hopf link.*

link with the additional trivial component, or to the connected sum of two Hopf links, or to the $(3, 3)$ -torus link, $\bar{6}_1^3$, represented by $(\sigma_1\sigma_2)^3$ (all linking numbers are equal to 1), or to the Borromean rings (represented by $(\sigma_1\sigma_2^{-1})^3$).

Proof: Our proof is based on the Coxeter theorem that the quotient group $B_3/(\sigma_i^4)$ is finite with 96 elements, [Cox]. Furthermore, $B_3/(\sigma_i^4)$ has 16 conjugacy classes¹⁴: 9 of them can be easily identified as representing trivial links (up to 4-move equivalence), and 2 of them represent the Hopf link ($\sigma_1^2\sigma_2$ and $\sigma_1^2\sigma_2^{-1}$), and σ_1^2 represents the Hopf link with an additional trivial component. We also have the connected sums of Hopf links ($\sigma_1^2\sigma_2^2$). Finally, we are left with two representatives of the link $\bar{6}_1^3$ ($\sigma_1\sigma_2^2\sigma_1\sigma_2^2$ and $\sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2^2$) and the Borromean rings. \square

Proposition 1.9 and Theorems 1.11, and 1.14 can be used to analyze 4-move equivalence classes of links with small number of crossings.

Theorem 1.15 (i) *Every knot of no more than 9 crossings is 4 move equivalent to the trivial knot.*

(ii) *Every two component link of no more than 9 crossings is 4-move equivalent to the trivial link of two components or to the Hopf link.*

Proof: Part (ii) follows immediately as the only 2-component links with up to 9 crossings which are not 2-algebraic are 9_{40}^2 , 9_{41}^2 , 9_{42}^2 and 9_{61}^2 and all these links are closed 3-braids. There are at most 6 knots with up to 9 crossings which are neither 2-algebraic nor 3-braid knots. They are: 9_{34} , 9_{39} , 9_{40} , 9_{41} , 9_{47} and 9_{49} . We reduced three of them, 9_{39} , 9_{41} and 9_{49} at my Fall 2003 Dean's Seminar. The knot 9_{40} was reduced in December of 2003 by Slavik Jablan and Radmila Sazdanovic. Soon after, my student Maciej Niebrzydowski simplified the remaining pair 9_{34} and 9_{47} , Fig.1.24. \square

¹⁴ $Id, \sigma_1, \sigma_1^{-1}, \sigma_1^2, \sigma_1\sigma_2, \sigma_1^{-1}\sigma_2, \sigma_1^{-1}\sigma_2^{-1}, \sigma_1^2\sigma_2, \sigma_1^2\sigma_2^{-1}, \sigma_1^2\sigma_2^2, \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}, \sigma_1\sigma_2^2\sigma_1\sigma_2^{-1}, \sigma_1\sigma_2^{-1}\sigma_1^2\sigma_2^{-1}, \sigma_1\sigma_2^2\sigma_1\sigma_2^2, \sigma_1^{-1}\sigma_2^2\sigma_1^{-1}\sigma_2^2, (\sigma_1\sigma_2^{-1})^3$ (checked by M. Dąbkowski using the *GAP* program).

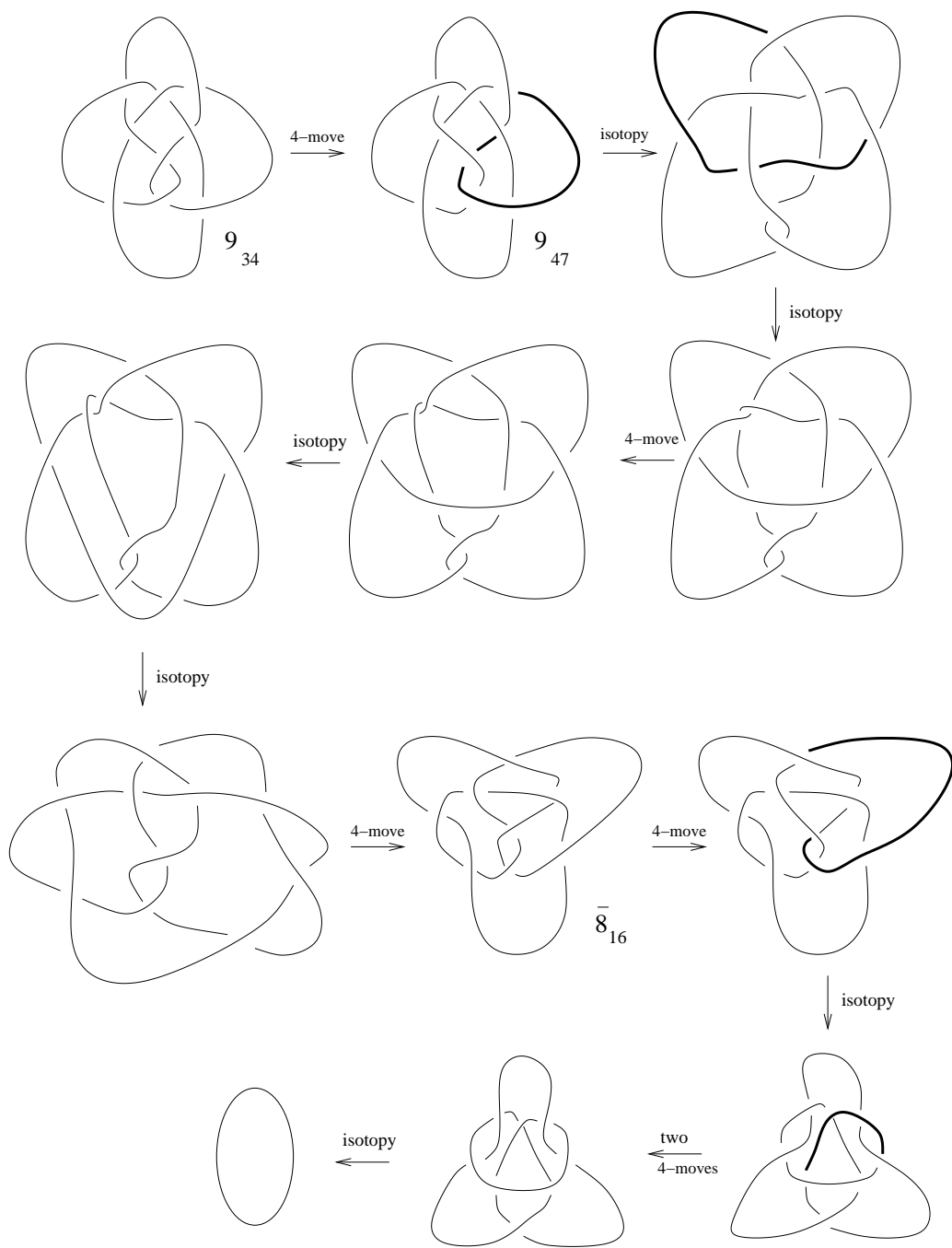


Fig. 1.24

A weaker version of the Kawauchi question has been answered by Nakanishi in 1989, [Nak-1]. If $\gamma \in B_n$ then the γ -move is the n -tangle move in which the trivial n -braid is replaced by the braid γ .

Theorem 1.16 (Nakanishi) *If two links L_1 and L_2 have the same linking matrix modulo 2, then L_2 can be obtained from L_1 by a finite number of ± 4 -moves and Δ_3^4 -moves.*

Proof: The square of the center, $\Delta_3^4 = (\sigma_1\sigma_2)^6$, of the 3-braid group B_3 and the Borromean braid, $(\sigma_1\sigma_2^{-1})^3$, are equal¹⁵ in $B_3/(\sigma_i^4)$. From this it also follows that Δ_3^4 and Δ_3^{-4} are equal in $B_3/(\sigma_i^4)$. Furthermore, the $(\sigma_1\sigma_2^{-1})^3$ -move is equivalent to Δ -move of Nakanishi in which $\sigma_1\sigma_2^{-1}\sigma_1$ is replaced by $\sigma_2\sigma_1^{-1}\sigma_2$ (we can think of this move as a “false” braid relation or a “false” third Reidemeister move). Nakanishi proved that two oriented links are Δ -move equivalent if and only if their linking matrices are equivalent [Nak-1]. Theorem 1.16 follows. \square

Selman Akbulut used Nakanishi’s theorem to prove John Nash’s conjecture for 3-dimensional manifolds [A-K]¹⁶.

It is not true that every link is 5-move equivalent to a trivial link. One can show, using the Jones polynomial, that the figure eight knot is not 5-move equivalent to any trivial link¹⁷. One can, however, introduce a more delicate

¹⁵We have in $B_3/(\sigma_i^4)$: $(\sigma_1\sigma_2)^6 = (\sigma_1^2\sigma_2\sigma_1^2\sigma_2)(\sigma_1\sigma_2^2\sigma_1\sigma_2^2) = \sigma_1^2\sigma_2\sigma_1^2(\sigma_1\sigma_2^2\sigma_1\sigma_2^2)\sigma_2 \stackrel{4}{=} \sigma_1^2\sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2^{-1} = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_2^2\sigma_1\sigma_2^{-1} \stackrel{4}{=} (\sigma_1\sigma_2^{-1})^3$. This calculation can be interpreted as an illustration of Fig.28 in [A-K].

¹⁶The conjecture that “any two closed smooth connected manifolds of the same dimension can be made diffeomorphic after blowing them up along submanifolds” is an interpretation of the Nash question “Is there an algebraic structure on a any given smooth manifold which is birational to RP^n ?” [Nash, A-K]. The conjecture is only loosely related to the question mentioned in the book “A beautiful mind” were in Chapter “The ‘Blowing Up’ Problem”, it is written: “Nash seemed, as the Fall [1963] unfolded, to be in far better shape than he had been during his previous interlude at the Institute [IAS]. As he said in his Madrid lecture, he “had had an idea which is referred to as Nash Blowing UP which I discussed with an eminent mathematician named Hironaka.” [Letter from J.Nash to V.Nash, 1.9.66] (Hironaka eventually wrote a conjecture up.)” [Nas].

¹⁷A 5-move preserves the absolute value of the Jones polynomial at $t = e^{\pi i/5}$ [P-1]. However, the Jones polynomial $V_{4_1}(e^{\pi i/5}) = 0$ but for any trivial link, T_n , we have $V_{T_n}(e^{\pi i/5}) = (-e^{\pi i/10} - e^{-\pi i/10})^{n-1} \neq 0$.

move, called $(2, 2)$ -move $(\infty\infty \rightarrow \text{X})$, such that the 5-move is a combination of a $(2, 2)$ -move and its mirror image $(-2, -2)$ -move $(\text{X} \rightarrow \infty\infty)$, as it is illustrated in Fig.1.25 [H-U, P-3].

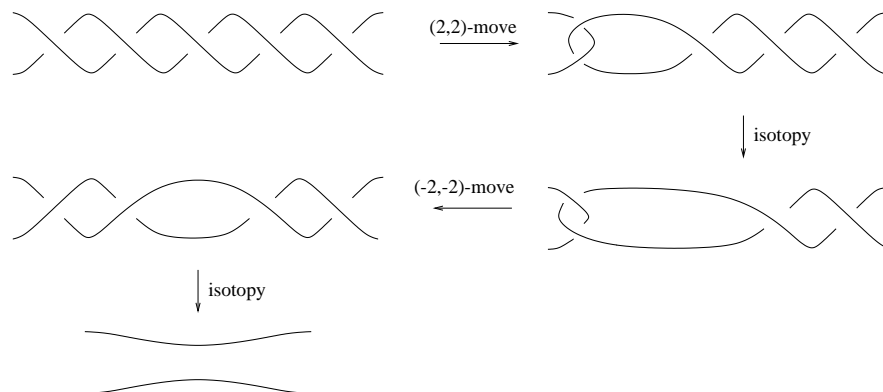


Fig. 1.25

Conjecture 1.17 (Harikae, Nakanishi, Uchida, 1992)

Every link is $(2, 2)$ -move equivalent to a trivial link.

As in the case of 3-moves, an elementary induction shows that the conjecture holds for 2-algebraic links. It is also known that the conjecture holds for all links up to 8 crossings. The key element of the argument in the proof is the observation (going back to Conway [Co]) that any link with up to 8 crossings (different from 8_{18} ; see footnote 19) is 2-algebraic. The reduction of the 8_{18} knot to the trivial link of two components by my students, Jarek Buczyński and Mike Veve, is illustrated in Fig.1.26.

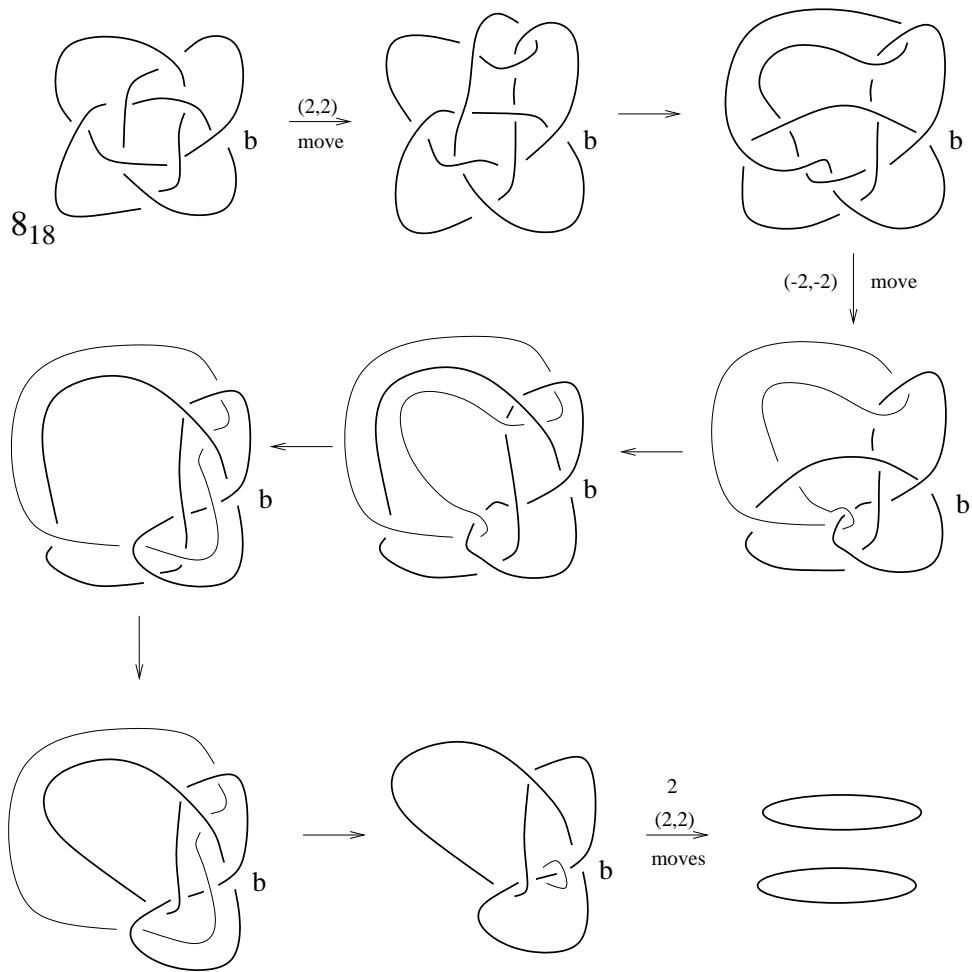


Fig. 1.26; Reduction of the 8_{18} knot

The smallest knots that are not reduced yet are 9_{40} and 9_{49} , Fig.1.27. Possibly you can reduce them!¹⁸

¹⁸We showed with M. Dąbkowski that the knots 9_{40} and 9_{49} are not $(2,2)$ -move equivalent to trivial links [D-P-2]. Possibly you can prove that they are in the same $(2,2)$ -move equivalence class! If I had to guess, I would say that it is a likely possibility.

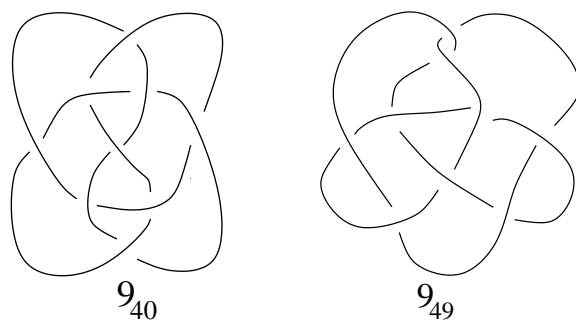


Fig. 1.27

I am much less convinced that the answer to the next open question is positive, so I will not call it a “conjecture”. First let us define a (p, q) -move to be a local modification of a link as shown in Fig.1.28. We say that two links, L_1 and L_2 , are (p, q) -equivalent if one can obtain one from the other by a finite number of (p, q) -, (q, p) -, $(-p, -q)$ - and $(-q, -p)$ -moves.

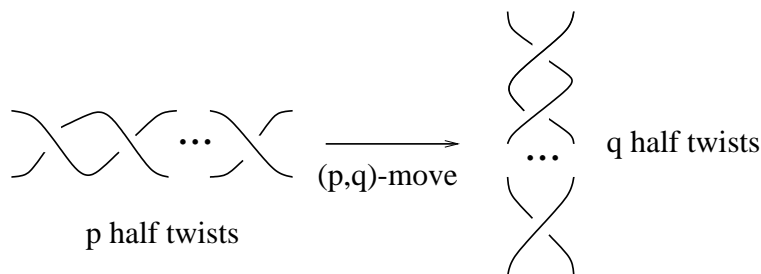


Fig. 1.28

Problem 1.18 ([Kir]; Problem 1.59(7), 1995) *Is it true that any link is $(2, 3)$ -move equivalent to a trivial link?*

Example 1.19 *Reduction of the trefoil and the figure eight knots is illustrated in Fig.1.29. Reduction of the Borromean rings is shown in Fig.1.30.*

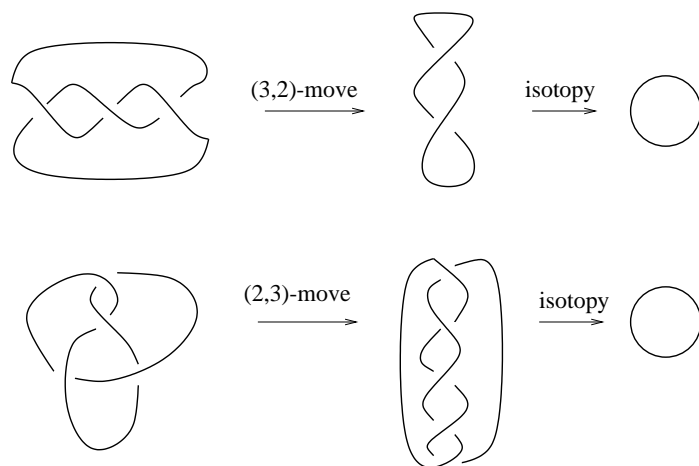


Fig. 1.29

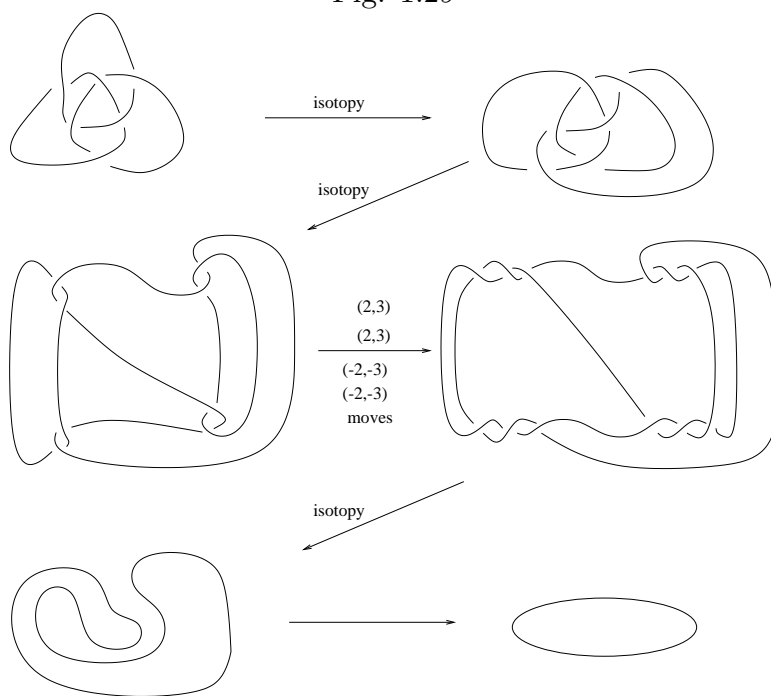


Fig. 1.30

As in the case of Proposition 1.9, simple inductive argument shows that 2-algebraic links are $(2,3)$ -move equivalent to trivial links. Fig.1.31 illustrates why the Borromean rings are 2-algebraic. By a proper filling of black

dots one can also show that all links with up to 8 crossings, except 8_{18} , are 2-algebraic. Thus, as in the case of $(2, 2)$ -equivalence, the only link with up to 8 crossings which still should be checked is the 8_{18} knot¹⁹. Nobody really worked on this problem seriously, so maybe somebody in the audience will try this puzzle.

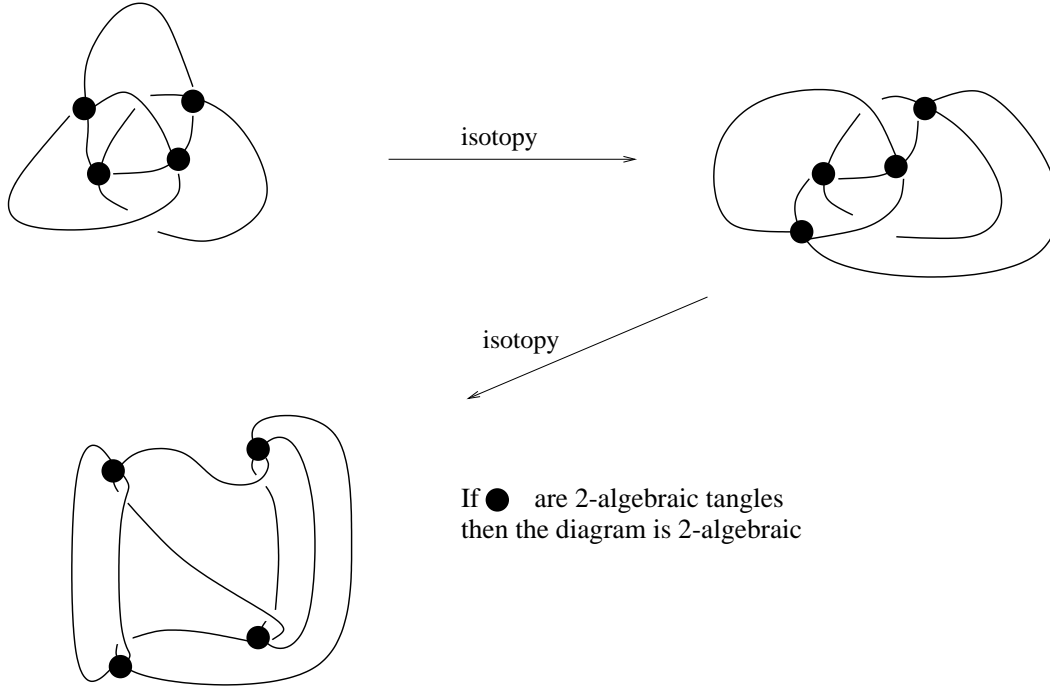


Fig. 1.31

¹⁹To prove that the knot 8_{18} is not 2-algebraic one considers the 2-fold branched cover of S^3 branched along the knot, $M_{8_{18}}^{(2)}$. Montesinos proved that algebraic knots are covered by Waldhausen graph manifolds [Mo-1]. Bonahon and Siebenmann showed ([B-S], Chapter 5) that $M_{8_{18}}^{(2)}$ is a hyperbolic 3-manifold so it cannot be a graph manifold. This manifold is interesting from the point of view of hyperbolic geometry because it is a closed manifold with its volume equal to the volume of the complement of figure eight knot [M-V-1]. The knot 9_{49} of Fig.1.27 is not 2-algebraic either because its 2-fold branched cover is a hyperbolic 3-manifold. In fact, it is the manifold I suspected from 1983 to have the smallest volume among oriented hyperbolic 3-manifolds [I-MPT, Kir, M-V-2]. In February of 2002 we (my student M.Dąbkowski and myself) found unexpected connection between Knot Theory and the theory of Burnside groups. This has allowed us to present simple combinatorial proof that the knots 9_{40} and 9_{49} are not 2-algebraic. However, our method does not work for the knot 8_{18} [D-P-1, D-P-2, D-P-3].

Fox colorings

The 3-coloring invariant which we are going to use to show that different trivial links are not 3-move equivalent, was introduced by R. H. Fox²⁰ around 1956 when he was explaining Knot Theory to undergraduate students at Haverford College (“in an attempt to make the subject accessible to everyone” [C-F]). It is a pleasant method of coding representations of the fundamental group of a link complement into the group of symmetries of an equilateral triangle, however this interpretation is not needed for the definition and most of applications of 3-colorings (compare [Cr, C-F, Fo-1, Fo-2]).

Definition 1.20 (*Fox 3-coloring of a link diagram*).

Consider a coloring of a link diagram using colors r (red), y (yellow), and b (blue) in such a way that an arc of the diagram (from an undercrossing to an undercrossing) is colored by one color and at each crossing one uses either only one or all three colors. Such a coloring is called a Fox 3-coloring. If the whole diagram is colored by just one color we say that we have a trivial coloring. The number of different Fox 3-colorings of D is denoted by $\text{tri}(D)$.

Example 1.21 (i) $\text{tri}(\bigcirc) = 3$ as the trivial link diagram has only trivial colorings.

(ii) $\text{tri}(\bigcirc \bigcirc) = 9$, and more generally, for the trivial link diagram of n components, T_n , one has $\text{tri}(T_n) = 3^n$.

²⁰Ralph Hartzler Fox was born March 24, 1913. A native of Morrisville, Pa., he attended Swarthmore College for two years while studying piano at the Leefson Conservatory of Music in Philadelphia. He was mostly home schooled and later he was a witness in a court case in Virginia, certifying soundness of home schooling. He received his master’s degree from the Johns Hopkins University and his Ph.D. from the Princeton University in 1939 under the supervision of Solomon Lefschetz. Fox was married, when he was still a student, to Cynthia Atkinson. They had one son, Robin. After receiving his Princeton doctorate, he spent the following year at Institute for Advanced Study in Princeton. He taught at the University of Illinois and Syracuse University before returning to join the Princeton University faculty in 1945 and staying there until his death. He was giving a series of lectures at the Instituto de Matemáticas de la Universidad Nacional Autónoma de México in the summer of 1951. He was lecturing to American Mathematical Society (1949), to the Summer Seminar of the Canadian Mathematical Society (1953), and at the Universities of Delft and Stockholm, while on a Fulbright grant (1952). He died December 23, 1973 in the University of Pennsylvania Graduate Hospital, where he had undergone open-heart surgery [P-12].

- (iii) For the standard diagram of the right-handed trefoil knot we have three trivial colorings and six nontrivial colorings. One of them is presented in Fig.1.32 (all the others differ from this one by permutations of colors). Thus, $tri(\langle \diamond \rangle) = 3 + 6 = 9$.

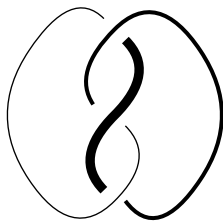


Fig. 1.32; Different colors are marked by lines of different thickness.

Fox 3-colorings were defined for link diagrams. They are, however, invariants of links. One only needs to show that $tri(D)$ is unchanged by Reidemeister moves.

The invariance under R_1 and R_2 is illustrated in Fig.1.33 and the invariance under R_3 is illustrated in Fig.1.34.

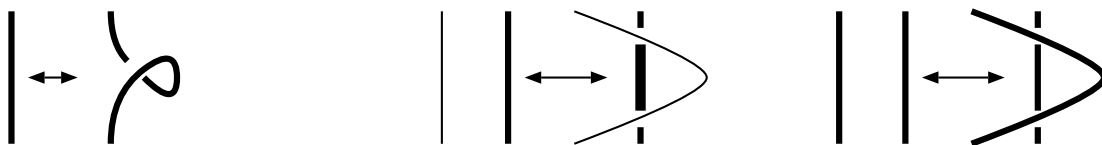


Fig. 1.33

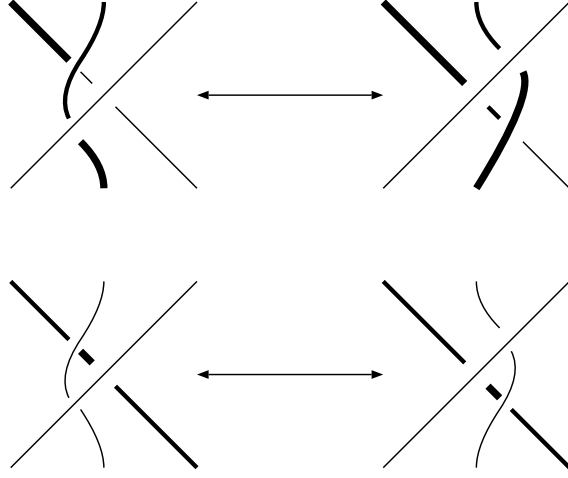


Fig. 1.34

The next property of Fox 3-colorings is the key in proving that different trivial links are not 3-move equivalent.

Lemma 1.22 ([P-1]) *3-moves do not change $\text{tri}(D)$.*

The proof of the lemma is illustrated in Figure 1.35.



Fig. 1.35

The lemma also explains the fact that the trefoil knot has nontrivial Fox 3-colorings: the trefoil knot is 3-move equivalent to the trivial link of two components (Example 1.4(i)).

Tomorrow, I will place the theory of Fox colorings in a more general (sophisticated) context, and apply it to the analysis of 3-moves (and $(2, 2)$ - and $(2, 3)$ -moves) on n -tangles. Interpretation of tangle colorings as Lagrangians in symplectic spaces is our main (and new) tool. In the third section, I will discuss another motivation for studying 3-moves: understanding skein modules based on their deformation.

2 Lagrangian approximation of Fox p -colorings of tangles

We just had the opportunity to listen to a beautiful and elementary talk by Lou Kauffman. I hope to follow this example by making my talk elementary and deep at the same time. I will use several results introduced by Lou, like classification of rational tangles, and also I am going to build on my yesterday's talk. I will culminate today's talk with introduction of the symplectic structure on the boundary of a tangle in such a way that tangles will yield Lagrangians in the associated symplectic space. I could not dream of this connection a year ago; however, now, 10 months after, I see the symplectic structure as a natural development.

Let us start our discussion slowly using my personal perspective and motivation. In the Spring of 1986, I was analyzing behavior of Jones type invariants of links when modified by k -moves (or t_k -, \bar{t}_{2k} -moves in the oriented case). My interest had its roots in the fundamental paper by Conway [Co]. In July of 1986, I gave a talk at the "Braids" conference in Santa Cruz. After my talk, I was told by Kunio Murasugi and Hitoshi Murakami about the Nakanishi's 3-move conjecture. It was suggested to me by R. Campbell (Rob Kirby's student in 1986) to consider the effect of 3-moves on Fox colorings. Several years later, when writing [P-3] in 1993, I realized that Fox colorings can be successfully used to analyze moves on tangles by considering not only the space of colorings but also the induced colorings of boundary points. More of this later, but let us now define Fox k -colorings first.

- Definition 2.1** (i) *We say that a link (or a tangle) diagram is k -colored if every arc is colored by one of the numbers $0, 1, \dots, k-1$ (elements of the group \mathbb{Z}_k) in such a way that at each crossing the sum of the colors of the undercrossings equals twice the color of the overcrossing modulo k ; see Fig.2.1.*
- (ii) *The set of k -colorings forms an abelian group, denoted by $Col_k(D)$ (we can also think of $Col_k(D)$ as a module over \mathbb{Z}_k). The cardinality of the group will be denoted by $col_k(D)$. For an n -tangle T each Fox k -coloring of T yields a coloring of boundary points of T and we have the homomorphism $\psi : Col_k(T) \rightarrow \mathbb{Z}_k^{2n}$*

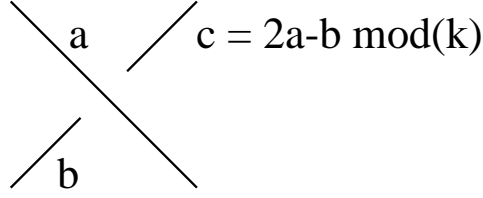


Fig. 2.1

It is a pleasant exercise to show that $Col_k(D)$ is unchanged by Reidemeister moves, so I am going to leave it for you. The invariance under k -moves is explained in Fig.2.2.

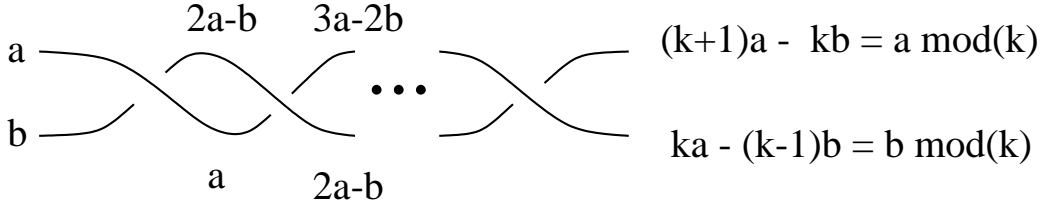


Fig. 2.2

Having observed that k -moves preserve the space of Fox k -colorings, let us take a closer look at the unlinking conjectures described before. We discussed the 3-move conjecture, the 4-move conjecture for knots, and the Kawauchi's question for links. As I mentioned yesterday, not every link can be simplified using 5-moves, but the 5-move is a combination of $\pm(2, 2)$ -moves and these moves might be sufficient to reduce every link to trivial links. Similarly not every link can be reduced via 7-moves, but again each 7-move is a combination of $(2, 3)$ -moves²¹ which still might be sufficient for reduction. We stopped at this point yesterday, but what could be used instead of general k -moves? Let us consider the case of p -moves, where p is a prime number. I suggest (and state publicly for the first time) that possibly one should consider *rational moves* instead, that is, moves in which a rational $\frac{p}{q}$ -tangle of Conway is substituted in place of the identity tangle²². The most important

²¹To be precise, a 7-move is a combination of $(-3, -2)$ - and $(2, 3)$ -moves; compare Fig.1.25.

²²The move was first considered by J. M. Montesinos [Mo-2]; compare also Y. Uchida [Uch].

observation for us is that $Col_p(D)$ is preserved by $\frac{p}{q}$ -moves. Fig.2.3 illustrates, for example, the fact that $Col_{13}(D)$ is unchanged by a $\frac{13}{5}$ -move.

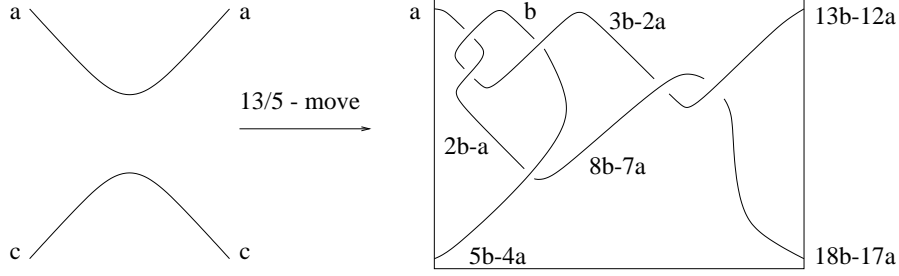


Fig. 2.3

We also should note that (m, q) -moves are equivalent to $\frac{mq+1}{q}$ -moves (Fig.2.4) so the space of Fox $(mq+1)$ -colorings is preserved by them too.

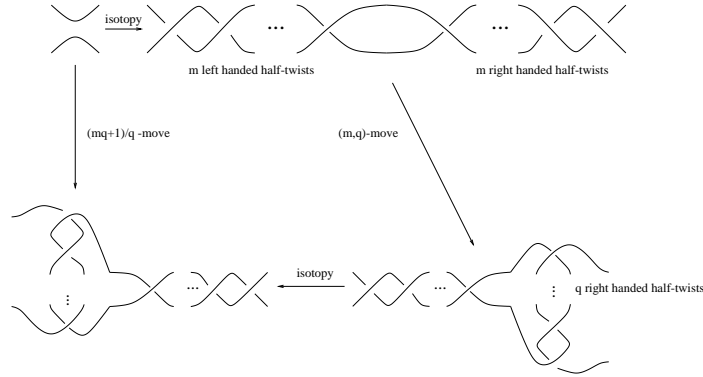


Fig. 2.4

We have just heard about Conway's classification of rational tangles in Lou's talk²³, so I will just briefly sketch necessary definitions and introduce basic notation. The 2-tangles shown in Fig.2.5 are called rational tangles – in Conway's notation, $T(a_1, a_2, \dots, a_n)$. A rational tangle is $\frac{p}{q}$ -tangle if $\frac{p}{q} = a_n + \frac{1}{a_{n-1} + \dots + \frac{1}{a_1}}$.²⁴ Conway proved that two rational tangles are ambient

²³L.Kauffman's talk in Cuautitlan, March, 2001; compare [K-L].

²⁴ $\frac{p}{q}$ is called the slope of the tangle and can be easily identified with the slope of the meridian disk of the solid torus being the branched double cover of the rational tangle.

isotopic (with boundary points fixed) if and only if their slopes are equal (compare [Kaw]).

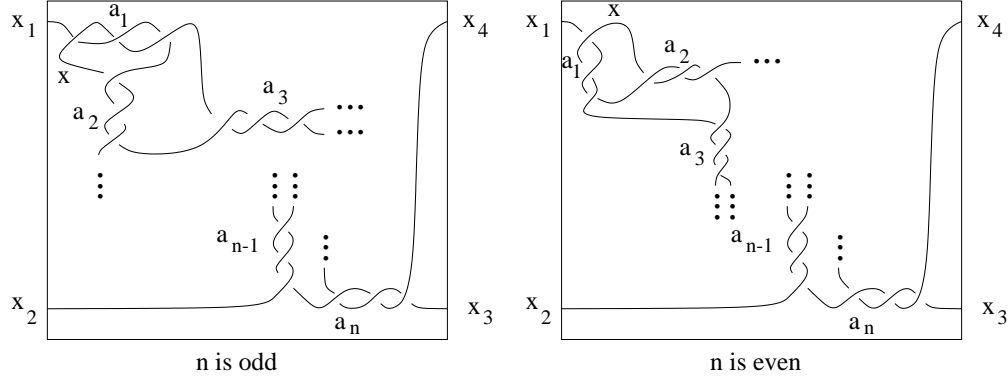


Fig. 2.5

For a given Fox coloring of the rational $\frac{p}{q}$ -tangle with boundary colors x_1, x_2, x_3, x_4 (Fig.2.5), one has $x_4 - x_1 = p(x - x_1)$, $x_2 - x_1 = q(x - x_1)$ and $x_3 = x_2 + x_4 - x_1$. If a coloring is nontrivial ($x_1 \neq x$) then $\frac{x_4 - x_1}{x_2 - x_1} = \frac{p}{q}$ as it has been explained by Lou.

Conjecture 2.2

Let p be a fixed prime number, then²⁵

- (i) Every link can be reduced to a trivial link by rational $\frac{p}{q}$ -moves (q any integer).
- (ii) There is a function $f(n, p)$ such that any n -tangle can be reduced to one of “basic” $f(n, p)$ n -tangles (allowing additional trivial components) by rational $\frac{p}{q}$ -moves.

First we observe that it suffices to use $\frac{p}{q}$ -moves with $|q| \leq \frac{p}{2}$, as they generate all the other $\frac{p}{q}$ -moves follow. Namely, we have $\frac{p}{p-q} = 1 + \frac{1}{-1+\frac{p}{q}}$ and $\frac{p}{-(p+q)} = -1 + \frac{1}{1+\frac{p}{q}}$. Thus $\frac{p}{q}$ -moves generate $\frac{p}{-q \pm p}$ -moves (e.g., $\frac{p}{p-q}$ tangle is reduced by an inverse of a $\frac{p}{q}$ -move to the 0-tangle, $1 + \frac{1}{-1+0} = 0$). Furthermore, we know

²⁵I decided to keep the word “Conjecture” as it was used in my talk. However, in Spring of 2002, we disproved it for any p , [D-P-1, D-P-2, D-P-3]. The talks in Mexico and Canada were essential for clarifying ideas and finally in constructing counterexamples.

that for odd p the $\frac{p}{1}$ -move is a combination of $\frac{p}{2}$ and $\frac{p}{-2}$ -moves (compare Fig.1.25). Thus, in fact, 3-move, (2, 2)-move and (2, 3)-move conjectures are special cases of Conjecture 2.2(i). For $p = 11$ we have $\frac{11}{2} = 5 + \frac{1}{2}$, $\frac{11}{3} = 4 - \frac{1}{3}$, $\frac{11}{4} = 3 - \frac{1}{4}$, $\frac{11}{5} = 2 + \frac{1}{5}$. Thus:

Conjecture 2.3

Every link can be reduced to a trivial link (with the same space of 11-colorings) by (2, 5)- and (4, -3)-moves, their inverses and their mirror images²⁶.

What about the number $f(n, p)$? We know that because $\frac{p}{q}$ -moves preserve p -colorings, therefore $f(n, p)$ is bounded from below by the number of subspaces of p -colorings of the $2n$ boundary points induced by Fox p -colorings of n -tangles (that is by the number of subspaces $\psi(\text{Col}_p(T))$ in \mathbb{Z}_p^{2n}). I noted in [P-3] that for 2-tangles this number is equal to $p + 1$ (even in this special case my argument was complicated). For $p = 3$ and $n = 4$ the number of subspaces followed from the work of my student Tatsuya Tsukamoto and is equal to 40 [P-Ts]. The combined effort of Mietek Dąbkowski and Tsukamoto gave the number 1120 for subspaces $\psi(\text{Col}_3(T))$ and 4-tangles. That was my knowledge at the early Spring of 2000. On May 2nd and 3rd I attended talks on Tits buildings (at the Banach Center in Warsaw) by Janek Dymara and Tadek Januszkiewicz. I realized that the topic may have some connection to my work. I asked Januszkiewicz whether he sees relation and I gave him numbers 4, 40, 1120 for $p = 3$. He immediately answered that most likely I was counting the number of Lagrangians in \mathbb{Z}_3^{2n-2} symplectic space, and that the number of Lagrangians in \mathbb{Z}_p^{2n-2} is known to be equal to $\prod_{i=1}^{n-1}(p^i + 1)$. Soon I constructed the appropriate symplectic form (as did Dymara). I will spend most of this talk on this construction and end with discussion of classes of tangles for which it has been proved that $f(n, p) = \prod_{i=1}^{n-1}(p^i + 1)$.

Consider $2n$ points on a circle (or a square) and a field \mathbb{Z}_p of p -colorings of a point. The colorings of $2n$ points form the linear space \mathbb{Z}_p^{2n} . Let e_1, \dots, e_{2n} be the standard basis of \mathbb{Z}_p^{2n} ,

²⁶As mentioned in the footnote 25, Conjecture 2.3 does not hold. The closure of the 3-braid Δ_3^4 provides the simplest counterexample [D-P-2, D-P-3]. However, it holds for 2-algebraic links; see Proposition 2.7.

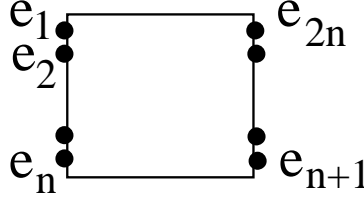


Fig. 2.6

$e_i = (0, \dots, 1, \dots, 0)$, where 1 occurs in the i -th position. Let $\mathbb{Z}_p^{2n-1} \subset \mathbb{Z}_p^{2n}$ be the subspace of vectors $\sum a_i e_i$ satisfying $\sum (-1)^i a_i = 0$ (alternating condition). Consider the basis f_1, \dots, f_{2n-1} of \mathbb{Z}_p^{2n-1} where $f_k = e_k + e_{k+1}$.

Let

$$\phi = \begin{pmatrix} 0 & 1 & \dots & \dots \\ -1 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & -1 & 0 \end{pmatrix}$$

be a skew-symmetric form ϕ on \mathbb{Z}_p^{2n-1} of nullity 1, that is,

$$\phi(f_i, f_j) = \begin{cases} 0 & \text{if } |j - i| \neq 1 \\ 1 & \text{if } j = i + 1 \\ -1 & \text{if } j = i - 1. \end{cases}$$

Notice that the vector $e_1 + e_2 + \dots + e_{2n}$ ($= f_1 + f_3 + \dots + f_{2n-1} = f_2 + f_4 + \dots + f_{2n}$) is ϕ -orthogonal to any other vector. If we consider $\mathbb{Z}_p^{2n-2} = \mathbb{Z}_p^{2n-1} / \mathbb{Z}_p$, where the subspace \mathbb{Z}_p is generated by $e_1 + \dots + e_{2n}$, that is, \mathbb{Z}_p consists of monochromatic (i.e., trivial) colorings, then ϕ descends to the symplectic form $\hat{\phi}$ on \mathbb{Z}_p^{2n-2} . Now we can analyze isotropic subspaces of

$(\mathbb{Z}_p^{2n-2}, \hat{\phi})$, that is subspaces on which $\hat{\phi}$ is 0 ($W \subset \mathbb{Z}_p^{2n-2}$, where $\hat{\phi}(w_1, w_2) = 0$ for all $w_1, w_2 \in W$). The maximal isotropic subspaces of \mathbb{Z}_p^{2n-2} are $(n-1)$ -dimensional and they are called Lagrangian subspaces (or maximal totally degenerated subspaces). There are $\prod_{i=1}^{n-1} (p^i + 1)$ of them.

Our local condition on Fox colorings (Fig.2.1) guarantees that for any tangle T , $\psi(\text{Col}_p T) \subset \mathbb{Z}_p^{2n-1}$. Furthermore, the space of trivial colorings, \mathbb{Z}_p is always in $\text{Col}_p T$. Thus ψ descends to $\hat{\psi} : \text{Col}_p T / \mathbb{Z}_p \rightarrow \mathbb{Z}_p^{2n-2} = \mathbb{Z}_p^{2n-1} / \mathbb{Z}_p$. Now we answer the fundamental question: Which subspaces of \mathbb{Z}_p^{2n-2} are yielded by n -tangles?

Theorem 2.4

$\hat{\psi}(\text{Col}_p T / \mathbb{Z}_p)$ is a Lagrangian subspace of \mathbb{Z}_p^{2n-2} with the symplectic form $\hat{\phi}$.

The natural question is whether every Lagrangian subspace can be realized as a space of induced colorings on the boundary for some tangle. The answer is negative for $p = 2$ and positive for $p > 2$.

Theorem 2.5 ([D-J-P])

- (i) For an odd prime number p , every Lagrangian in $(\mathbb{Z}_p^{2n-2}, \hat{\phi})$ is realized as $\hat{\psi}(\text{Col}_p T / \mathbb{Z}_p)$ for some n -tangle T . Furthermore, T can be chosen to be a rational n -tangle²⁷.
- (ii) For $p = 2$, $n > 2$, not every Lagrangian is realized as $\hat{\psi}(\text{Col}_2 T / \mathbb{Z}_2)$. We have $f(n, 2) = \prod_{i=1}^{n-1} (2i + 1)$ (a 2-coloring is unchanged by a crossing change) but the number of Lagrangians is equal to $\prod_{i=1}^{n-1} (2^i + 1)$.

As a corollary we obtain a fact which was considered to be difficult before, even for 2-tangles (compare [P-3, J-P]).

Corollary 2.6

For any p -coloring \mathbf{x} of a tangle boundary points satisfying the alternating property (i.e., $\mathbf{x} \in \mathbb{Z}_p^{2n-1}$) there is an n -tangle and p -coloring of it that yields \mathbf{x} . In other words: $\mathbb{Z}_p^{2n-1} = \bigcup_T \psi_T(\text{Col}_p T)$. Furthermore, the space $\psi_T(\text{Col}_p T)$ is n -dimensional for any T .

²⁷An n -tangle is a rational (or n -bridge) tangle if it is homeomorphic to a tangle without crossing and trivial components (we allow homeomorphism moving the boundary of the 3-ball). Alternatively, we can use an inductive definition modifying Definition 2.9 in such a way that we start from a tangle without a crossing and a trivial component and we assume in condition (i)(1) that B has exactly one crossing (which is not nugatory, that is, it cannot be eliminated by a first Reidemeister move).

We can say that we understand the lower bound for the function $f(n, p)$, but when does Conjecture 2.2 holds with $f(n, p) = \prod_{i=1}^{n-1} (p^i + 1)$?

In [D-I-P] we discuss Conjecture 2.2 for 2-algebraic tangles. Here we sketch a proof of a simpler fact.

Proposition 2.7

Let p be a fixed prime number and let be H_p the family of 2-tangles: $\frac{1-p}{2}, \frac{3-p}{2}, \dots, 0, \dots, \frac{p-3}{2}, \frac{p-1}{2}$ and ∞ (horizontal family), and let V_p be the vertical family of 2-tangles, $V_p = r(H_p)$; then

- (i) Every 2-algebraic tangle can be reduced to a 2-tangle from the family H_p (resp. V_p) with possible additional trivial components by $(\frac{sp}{q})$ -moves, where s and q are any integers such that sp and q are relatively prime. Furthermore, for $p \leq 13$ one can assume that $s = 1$.
- (ii) Every 2-algebraic link can be reduced to a trivial link by $(\frac{sp}{q})$ -moves, where s and q are any integers such that sp and q are relatively prime. Furthermore, for $p \leq 13$ one can assume that $s = 1$.

Outline of the proof. We use the structure of 2-algebraic tangles to perform an inductive proof similar to that of Proposition 1.9. The main problem in the proof is to show that the family V_p can be reduced to the family H_p by our moves. Consider the vertical tangle $r(k)$ where k is relatively prime to p . There are integers k' and s such that $kk' + 1 = sp$ or equivalently $k' + \frac{1}{k} = \frac{sp}{k}$. Therefore the $\frac{sp}{k}$ -move (equivalently (k, k') -move) is changing $r(k)$ to the horizontal tangle k' . In this reasoning we do not have a control over s . Consider now the case of $p = 13$ and $s = 1$. By considering fractions $\frac{13}{2} = 6 + \frac{1}{2}$, $\frac{13}{3} = 4 + \frac{1}{3}$, $\frac{13}{4} = 3 + \frac{1}{4}$, $\frac{13}{6} = 2 + \frac{1}{6}$, we are able to work with all $r(k)$ except $k = 5$. $5 + \frac{1}{5} = \frac{26}{5}$ so $s = 2$ in this case. We can, however, realize $\frac{26}{5}$ -move as a combination of $\frac{13}{3}$ -move and $\frac{13}{2}$ -move as illustrated in Fig.2.7 (we start by presenting $\frac{26}{5}$ as $6 + \frac{1}{-1-\frac{1}{4}}$).

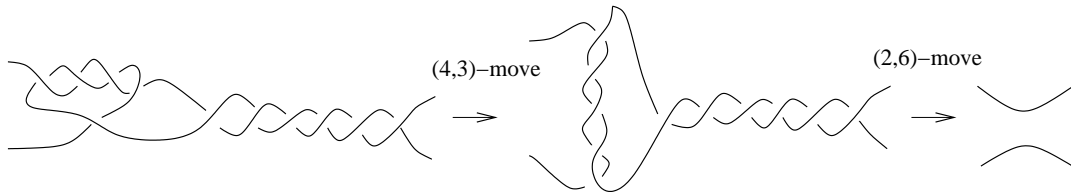


Fig. 2.7

Corollary 2.8

Every $(\frac{sp}{q})$ -rational tangle, p odd prime, can be reduced to the 0 2-tangle by (k, k') -moves where $|k| < \frac{p}{2}$ and $kk' + 1 = sp$ for some s .

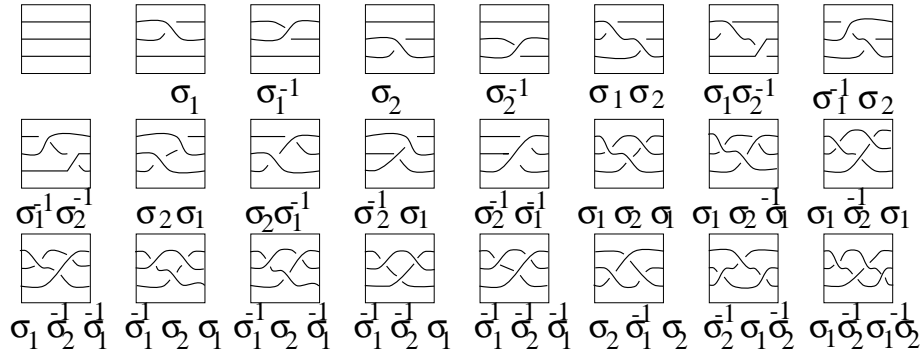
In order to be able to use induction for n -tangles with $n > 2$, we generalize the notion of the algebraic tangle.

Definition 2.9

- (i) *The family of n -algebraic tangles is the smallest family of n -tangles which satisfies:*
 - (0) *Any n -tangle with 0 or 1 crossing is n -algebraic.*
 - (1) *If A and B are n -algebraic tangles then $r^i(A) * r^j(B)$ is n -algebraic, where r denotes here the rotation of a tangle by $\frac{2\pi}{2n}$ angle, and $*$ denotes horizontal composition of tangles.*
- (ii) *If in the condition (1), B is restricted to tangles with no more than k crossings, we obtain the family of (n, k) -algebraic tangles.*
- (iii) *If an m -tangle, T , is obtained from an (n, k) -algebraic tangle (resp. n -algebraic tangle) by partially closing its endpoints ($2n - 2m$ of them) without introducing any new crossings, then T is called an (n, k) -algebraic (resp. n -algebraic) m -tangle. For $m = 0$ we obtain an (n, k) -algebraic (resp. n -algebraic) link.*

Conjecture 2.2, for $p = 3$, has been proved for 3-algebraic tangles [P-Ts] ($f(3, 3) = 40$) and $(4, 5)$ -algebraic tangles [Tsu] ($f(4, 3) = 1120$). In particular the Montesinos-Nakanishi 3-move conjecture holds for 3-algebraic and $(4, 5)$ -algebraic links. 40 “basic” 3-tangles are shown in Fig. 2.8.

Invertible (braid type) basic tangles



Noninvertible basic tangles

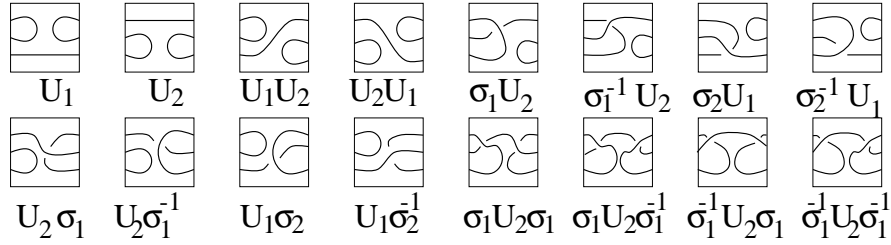


Fig. 2.8

The simplest 4-tangles which cannot be distinguished by 3-colorings for which 3-move equivalence is not yet established are illustrated in Fig.2.9. As for (2, 2)-moves, the equivalence of 2-tangles in Fig.2.10 is an open problem²⁸.

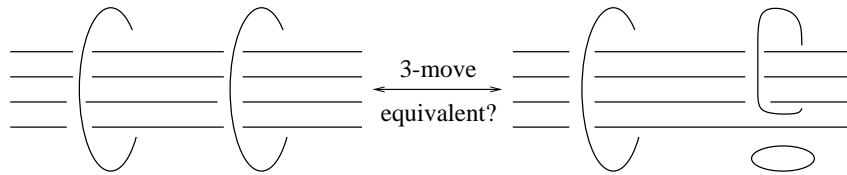


Fig. 2.9

²⁸The 4-tangles in Fig.2.9 are not 3-move equivalent. This follows from the fact that the Borromean rings and the Chen's link are not 3-move equivalent to trivial links [D-P-1, D-P-3]. Similarly, the fact that 2-tangles of Fig.2.10 are not (2, 2)-move equivalent follows from the result proven in [D-P-2, D-P-3] that the knot 9_{49} and the link 9_{40}^2 are not (2, 2)-move equivalent to the trivial link of three components.

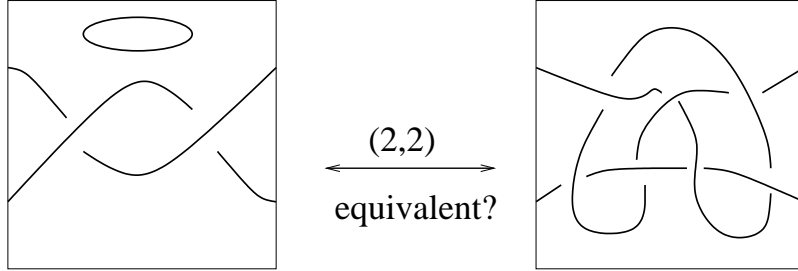


Fig. 2.10

A weaker version of the Montesinos-Nakanishi 3-move conjecture has been proved by Bronek Wajnryb in 1985 [Wa-1, Wa-2] (compare Theorem 1.16).

Theorem 2.10 (Wajnryb) *Every link can be reduced to a trivial link by a finite number of ± 3 -moves and Δ_5^4 -moves.*

Let me complete this talk by mentioning two generalizations of the Fox k -colorings.

In the first generalization we consider any commutative ring with the identity, \mathcal{R} , instead of \mathbb{Z}_k . We construct $Col_{\mathcal{R}}T$ in the same way as before with the relation at each crossing, Fig.2.1, having the form $c = 2a - b$ in \mathcal{R} . The skew-symmetric form ϕ on \mathcal{R}^{2n-1} , the symplectic form $\hat{\phi}$ on \mathcal{R}^{2n-2} and the homomorphisms ψ and $\hat{\psi}$ are defined in the same manner as before. Theorem 2.4 generalizes as follows ([D-J-P]):

Theorem 2.11 *Let \mathcal{R} be a Principal Ideal Domain (PID). Then, $\hat{\psi}(Col_{\mathcal{R}}T/\mathcal{R})$ is a virtual Lagrangian submodule of \mathcal{R}^{2n-2} with the symplectic form $\hat{\phi}$. That is, $\hat{\psi}(Col_{\mathcal{R}}T/\mathcal{R})$ is a finite index submodule of a Lagrangian in \mathcal{R}^{2n-2} .*

This result can be used to analyze embeddability of tangles in links. It gives an alternative proof of Theorem 2.2 in [P-S-W] in the case of the 2-fold cyclic cover of B^3 branched over a tangle.

Example 2.12 *Consider the pretzel tangle $T = (p, -p)$, Fig.2.11, and the ring $\mathcal{R} = \mathbb{Z}$. Then the virtual Lagrangian $\hat{\psi}(Col_{\mathbb{Z}}T/\mathbb{Z})$ has index p . Namely, coloring of T , as illustrated in Fig.2.11, forces us to have $a = b$ and modulo trivial colorings the image $\hat{\psi}(Col_{\mathbb{Z}}T/\mathbb{Z})$ is generated by the vector $(0, p, p, 0) = p(e_2 + e_3)$. The symplectic space $(\mathbb{Z}^{4-2}, \hat{\phi})$ has a basis $(e_1 + e_2, e_2 + e_3)$. Thus, $\hat{\psi}(Col_{\mathbb{Z}}T/\mathbb{Z})$ is a virtual Lagrangian of index p .*

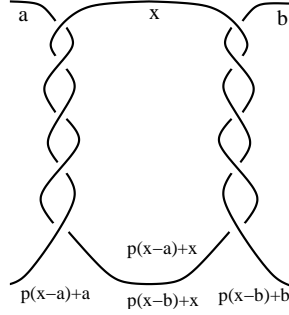


Fig. 2.11; $p = 5$

Corollary 2.13

If $\hat{\psi}(\text{Col}_\mathbb{Z} T / \mathbb{Z})$ is a virtual Lagrangian of index $p > 1$, then T does not embed in the trivial knot.

The second generalization leads to racks and quandles [Joy, F-R], but we restrict our remarks to the abelian case – Alexander-Burau-Fox colorings²⁹. An ABF-coloring uses colors from a ring \mathcal{R} with an invertible element t (e.g., $\mathcal{R} = \mathbb{Z}[t^{\pm 1}]$). The relation in Fig.2.1 is modified to the relation $c = (1-t)a + tb$ in \mathcal{R} at each crossing of an oriented link diagram; see Fig. 2.12.

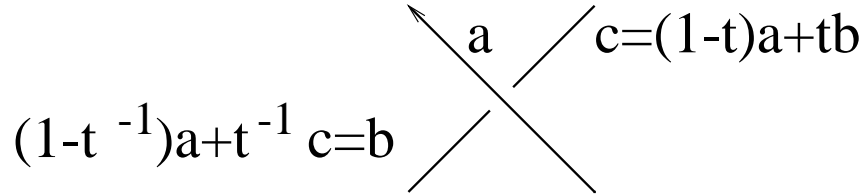


Fig. 2.12

²⁹The related approach was first outlined in the letter of J. W. Alexander to O. Veblen, 1919 [A-V]. Alexander was probably influenced by Poul Heegaard's dissertation, 1898, which he reviewed for the French translation [Heeg]. Burau was considering a braid representation, but locally his relation was the same as that of Fox. According to J. Birman, Burau learned about the representation from Reidemeister or Artin [Ep], p.330.

The space \mathcal{R}^{2n-2} has a natural Hermitian structure [Sq], but one can also find a symplectic structure and prove a version of Theorem 2.11 in this setting [D-J-P].

3 Historical Introduction to Skein Modules

In my last talk of the conference, I will discuss *skein modules*, or as I prefer to say more generally, *algebraic topology based on knots*. It was my mind's child, even if the idea was also conceived by other people (most notably Vladimir Turaev), and was envisioned by John H. Conway (as “linear skein”) a decade earlier. Skein modules have their origin in the observation made by Alexander [Al], that his polynomials (*Alexander polynomials*) of three links, L_+ , L_- and L_0 in R^3 are linearly related (Fig.3.2).

For me it started in Norman, Oklahoma in April of 1987, when I was enlightened to see that the multivariable version of the Jones-Conway (Homflypt) polynomial analyzed by Jim Hoste and Mark Kidwell is really a module of links in a solid torus (or more generally, in the connected sum of solid tori).

I would like to discuss today, in more detail, skein modules related to the (deformations) of 3-moves and the Montesinos-Nakanishi 3-move conjecture, but first I will give the general definition and I will make a short tour of the world of skein modules.

Skein Module is an algebraic object associated with a manifold, usually constructed as a formal linear combination of embedded (or immersed) sub-manifolds, modulo locally defined relations. In a more restricted setting a skein module is a module associated with a 3-dimensional manifold, by considering linear combinations of links in the manifold, modulo properly chosen (skein) relations. It is the main object of the *algebraic topology based on knots*. When choosing relations one takes into account several factors:

- (i) Is the module we obtain accessible (computable)?
- (ii) How precise are our modules in distinguishing 3-manifolds and links in them?
- (iii) Does the module reflect topology/geometry of a 3-manifold (e.g., surfaces in a manifold, geometric decomposition of a manifold)?
- (iv) Does the module admit some additional structure (e.g., filtration, gradation, multiplication, Hopf algebra structure)? Is it leading to a Topo-

logical Quantum Field Theory (TQFT) by taking a finite dimensional quotient?

One of the simplest skein modules is a q -deformation of the first homology group of a 3-manifold M , denoted by $\mathcal{S}_2(M; q)$. It is based on the skein relation (between oriented framed links in M): $L_+ = qL_0$; it also satisfies the framing relation $L^{(1)} - qL$, where $L^{(1)}$ denote a link obtained from L by twisting the framing of L once in the positive direction. This easily defined skein module “sees” already nonseparating surfaces in M . These surfaces are responsible for torsion part of our skein module [P-10].

There is a more general pattern: most of the analyzed skein modules reflect various surfaces embedded in a manifold.

The best studied skein modules use skein relations which worked successfully in the classical Knot Theory (when defining polynomial invariants of links in R^3).

- (1) The Kauffman bracket skein module, KBSM.

The skein module based on the *Kauffman bracket skein relation*, $L_+ = AL_- + A^{-1}L_\infty$, and denoted by $S_{2,\infty}(M)$, is the best understood among the Jones type skein modules. It can be interpreted as a quantization of the co-ordinate ring of the character variety of $SL(2, \mathbb{C})$ representations of the fundamental group of the manifold M , [Bu-2, B-F-K, P-S]. For $M = F \times [0, 1]$, KBSM is an algebra (usually noncommutative). It is finitely generated algebra for a compact F [Bu-1], and has no zero divisors [P-S]. The center of the algebra is generated by boundary components of F [B-P, P-S]. Incompressible tori and 2-spheres in M yield torsion in KBSM; it is the question of fundamental importance whether other surfaces could yield torsion as well. The Kauffman bracket skein modules of the exteriors of 2-bridge knots have been recently (April 2004) computed by Thang Le [Le]. For a 2-bridge (rational) knot $K_{\frac{p}{m}}$ the skein module is the free $\mathbb{Z}[A^{\pm 1}]$ module with the basis $\{x^i y^j\}$, $0 \leq i, 0 \leq j \leq \frac{p-1}{2}$, where $x^i y^j$ denotes the element of the skein module represented by the link composed of i parallel copies of the meridian curve x and j parallel copies of the curve y ; see Fig.3.1. Le’s theorem generalizes results in [Bu-3] and [B-L].

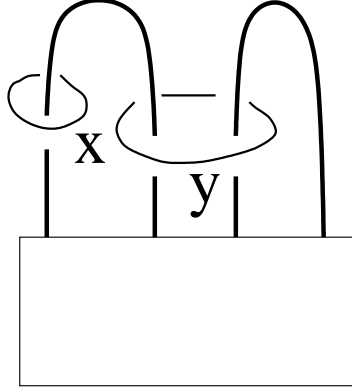


Fig. 3.1

- (2) Skein modules based on the Jones-Conway (Homflypt) relation.
 $v^{-1}L_+ - vL_- = zL_0$, where L_+, L_-, L_0 are oriented links (Fig.3.2).
 These skein modules are denoted by $S_3(M)$ and generalize skein modules based on Conway relation which were hinted at by Conway. For $M = F \times [0, 1]$, $S_3(M)$ is a *Hopf algebra* (usually neither commutative nor co-commutative), [Tu-2, P-6]. $S_3(F \times [0, 1])$ is a free module and can be interpreted as a quantization [H-K, Tu-1, P-5, Tu-2]. $S_3(M)$ is related to the algebraic set of $SL(n, \mathbb{C})$ representations of the fundamental group of the manifold M , [Si].

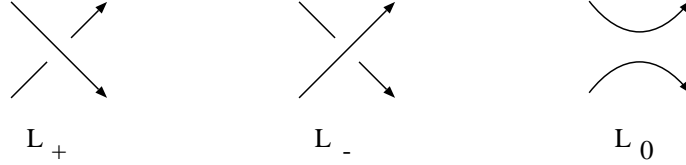


Fig. 3.2

- (3) Skein modules based on the *Kauffman polynomial* relation.
 $L_{+1} + L_{-1} = x(L_0 + L_\infty)$ (see Fig.3.3) and the framing relation $L^{(1)} - aL$.
 This module is denoted by $S_{3,\infty}$ and is known to be free for $M = F \times [0, 1]$.
- (4) Homotopy skein modules.
 In these skein modules, $L_+ = L_-$ for self-crossings. The best studied example is the q -homotopy skein module with the skein relation $q^{-1}L_+ - qL_- = zL_0$ for mixed crossings. For $M = F \times [0, 1]$ it is a quantization, [H-P-1, Tu-2, P-11], and as noted by Uwe Kaiser they

can be almost completely understood using singular tori technique introduced by Xiao-Song Lin.

- (5) Skein modules based on Vassiliev-Gusarov filtration.

We extend the family of knots, \mathcal{K} , by allowing singular knots, and resolve a singular crossing by $K_{cr} = K_+ - K_-$. These allow us to define the Vassiliev-Gusarov filtration: $\dots \subset C_3 \subset C_2 \subset C_1 \subset C_0 = R\mathcal{K}$, where C_k is generated by knots with k singular crossings. The k th Vassiliev-Gusarov skein module is defined to be a quotient: $W_k(M) = R\mathcal{K}/C_{k+1}$. The completion of the space of knots with respect to the Vassiliev-Gusarov filtration, $\hat{R}\mathcal{K}$, is a *Hopf algebra* (for $M = S^3$). Functions dual to Vassiliev-Gusarov skein modules are called *finite type* or *Vassiliev invariants* of knots, [P-7].

- (6) Skein modules based on relations deforming n -moves.

$\mathcal{S}_n(M) = R\mathcal{L}/(b_0L_0 + b_1L_1 + b_2L_2 + \dots + b_{n-1}L_{n-1})$. In the unoriented case, we can add to the relation the term $b_\infty L_\infty$ to get $\mathcal{S}_{n,\infty}(M)$, and also, possibly, a framing relation. The case $n = 4$, on which I am working with my students will be described, in greater detail in a moment.

Examples (1)-(5) gave a short description of skein modules studied extensively until now. I will now spend more time on two other examples which only recently have been considered in more detail. The first example is based on a deformation of the 3-move and the second on the deformation of the $(2, 2)$ -move. The first one has been studied by my students Tsukamoto and Veve. I denote the skein module described in this example by $\mathcal{S}_{4,\infty}$ since it involves (in the skein relation) 4 horizontal positions and the vertical (∞) smoothing.

Definition 3.1 *Let M be an oriented 3-manifold and let \mathcal{L}_{fr} be the set of unoriented framed links in M (including the empty link, \emptyset), and let R be any commutative ring with identity. Then we define the $(4, \infty)$ skein module as: $\mathcal{S}_{4,\infty}(M; R) = R\mathcal{L}_{fr}/I_{(4,\infty)}$, where $I_{(4,\infty)}$ is the submodule of $R\mathcal{L}_{fr}$ generated by the skein relation:*

$b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 + b_\infty L_\infty = 0$ and the framing relation:

$L^{(1)} = aL$ where a, b_0, b_3 are invertible elements in R and b_1, b_2, b_∞ are any fixed elements of R (see Fig.3.3).

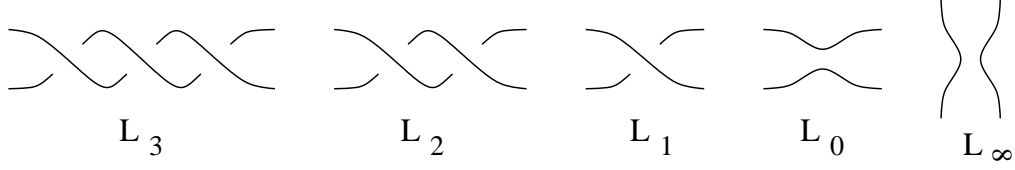


Fig. 3.3

The generalization of the Montesinos-Nakanishi 3-move conjecture says that $\mathcal{S}_{4,\infty}(S^3, R)$ is generated by trivial links³⁰ and that for n -tangles our skein module is generated by $f(n, 3)$ basic tangles (with possible trivial components). This would give a generating set for our skein module of S^3 or D^3 with $2n$ boundary points (an n -tangle). In [P-Ts] we analyzed extensively the possibility that trivial links, T_n , are linearly independent. This may happen if $b_\infty = 0$ and $b_0 b_1 = b_2 b_3$. These lead to the following conjecture:

Conjecture 3.2 (1) *There is a polynomial invariant of unoriented links in S^3 , $P_1(L) \in \mathbb{Z}[x, t]$, which satisfies:*

- (i) *Initial conditions: $P_1(T_n) = t^n$, where T_n is a trivial link of n components.*
- (ii) *Skein relation: $P_1(L_0) + xP_1(L_1) - xP_1(L_2) - P_1(L_3) = 0$, where L_0, L_1, L_2, L_3 is a standard, unoriented skein quadruple (L_{i+1} is obtained from L_i by a right-handed half-twist on two arcs involved in L_i ; compare Fig.3.3).*

(2) *There is a polynomial invariant of unoriented framed links, $P_2(L) \in \mathbb{Z}[A^{\pm 1}, t]$ which satisfies:*

- (i) *Initial conditions: $P_2(T_n) = t^n$,*
- (ii) *Framing relation: $P_2(L^{(1)}) = -A^3 P_2(L)$ where $L^{(1)}$ is obtained from a framed link L by a positive half twist on its framing.*
- (iii) *Skein relation: $P_2(L_0) + A(A^2 + A^{-2})P_2(L_1) + (A^2 + A^{-2})P_2(L_2) + AP_2(L_3) = 0$.*

³⁰The counterexamples to the Montesinos-Nakanishi 3-move conjecture, [D-P-1], can be used to show that trivial links “generically” do not generate $\mathcal{S}_{4,\infty}(S^3, R)$. This happens, for example, if there is a proper ideal $\mathcal{I} \in R$ such that b_1, b_2 and b_∞ are in \mathcal{I} .

The above conjectures assume that $b_\infty = 0$ in our skein relation. Let us consider, for a moment, the possibility that b_∞ is invertible in R . Using the “denominator” of our skein relation (Fig.3.4) we obtain the relation which allows us to compute the effect of adding a trivial component to a link L (we write t^n for the trivial link T_n):

$$(*) \quad (a^{-3}b_3 + a^{-2}b_2 + a^{-1}b_1 + b_0 + b_\infty t)L = 0$$

When considering the “numerator” of the relation and its mirror image (Fig.3.4) we obtain formulas for Hopf link summands, and because the un-oriented Hopf link is amphicheiral we can eliminate it from our equations to get the following formula (**):

$$b_3(L\#H) + (ab_2 + b_1t + a^{-1}b_0 + ab_\infty)L = 0.$$

$$b_0(L\#H) + (a^{-1}b_1 + b_2t + ab_3 + a^2b_\infty)L = 0.$$

$$(**) \quad ((b_0b_1 - b_2b_3)t + (a^{-1}b_0^2 - ab_3^2) + (ab_0b_2 - a^{-1}b_1b_3) + b_\infty(ab_0 - a^2b_3))L = 0.$$

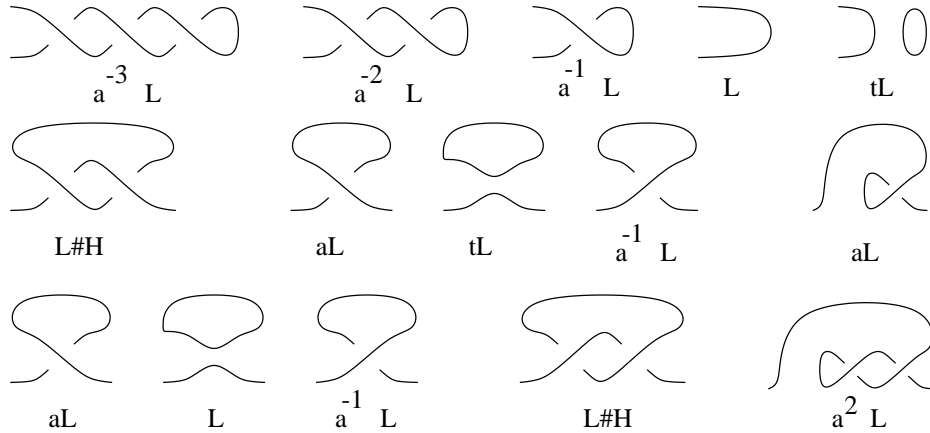


Fig. 3.4

It is possible that (*) and (**) are the only relations in the module. More precisely, we ask whether $\mathcal{S}_{4,\infty}(S^3; R)$ is the quotient ring $R[t]/(\mathcal{I})$ where t^i represents the trivial link of i components and \mathcal{I} is the ideal generated by (*) and (**) for $L = t$. The interesting substitution which satisfies the relations is $b_0 = b_3 = a = 1$, $b_1 = b_2 = x$, $b_\infty = y$. This may lead to a new polynomial

invariant (in $\mathbb{Z}[x, y]$) of unoriented links in S^3 satisfying the skein relation $L_3 + xL_2 + xL_1 + L_0 + yL_\infty = 0$.³¹

What about the relations to the Fox colorings? One such a relation, that was already mentioned, is the use of 3-colorings to estimate the number of basic n -tangles (by $\prod_{i=1}^{n-1} (3^i + 1)$) for the skein module $\mathcal{S}_{4,\infty}$. I am also convinced that $\mathcal{S}_{4,\infty}(S^3; R)$ contains full information about the space of Fox 7-colorings. It would be a generalization of the fact that the Kauffman bracket polynomial contains information about 3-colorings and the Kauffman polynomial contains information about 5-colorings. In fact, François Jaeger told me that he knew how to form a short skein relation (of the type $(\frac{p+1}{2}, \infty)$) involving spaces of p -colorings. Unfortunately, François died prematurely in 1997 and I do not know how to prove his statement³².

Finally, let me shortly describe the skein module related to the deformation of $(2, 2)$ -moves. Because a $(2, 2)$ -move is equivalent to the rational $\frac{5}{2}$ -move, I will denote the skein module by $\mathcal{S}_{\frac{5}{2}}(M; R)$.

Definition 3.3 *Let M be an oriented 3-manifold. Let \mathcal{L}_{fr} be the set of unoriented framed links in M (including the empty link, \emptyset) and let R be any commutative ring with identity. We define the $\frac{5}{2}$ -skein module as $\mathcal{S}_{\frac{5}{2}}(M; R) = R\mathcal{L}_{fr}/(I_{\frac{5}{2}})$ where $I_{\frac{5}{2}}$ is the submodule of $R\mathcal{L}_{fr}$ generated by the skein relation:*

$$(i) \quad b_2L_2 + b_1L_1 + b_0L_0 + b_\infty L_\infty + b_{-1}L_{-1} + b_{-\frac{1}{2}}L_{-\frac{1}{2}} = 0,$$

its mirror image:

$$(i') \quad b'_2L_2 + b'_1L_1 + b'_0L_0 + b'_\infty L_\infty + b'_{-1}L_{-1} + b'_{-\frac{1}{2}}L_{-\frac{1}{2}} = 0$$

and the framing relation:

$L^{(1)} = aL$, where $a, b_2, b'_2, b_{-\frac{1}{2}}, b'_{-\frac{1}{2}}$ are invertible elements in R and $b_1, b'_1, b_0, b'_0, b_{-1}, b'_{-1}, b_\infty$, and b'_∞ are any fixed elements of R . The links $L_2, L_1, L_0, L_\infty, L_{-1}, L_{\frac{1}{2}}$ and $L_{-\frac{1}{2}}$ are illustrated in Fig.3.5.³³

³¹This speculation should be modified keeping in mind the fact that the Montesinos-Nakanishi 3-move conjecture does not hold [D-P-1].

³²If $col_p(L) = |Col_p(L)|$ denotes the order of the space of Fox p -colorings of the link L , then among $p + 1$ links L_0, L_1, \dots, L_{p-1} , and L_∞ , p of them has the same order $col_p(L)$ and one has its order p times larger [P-3]. This leads to the relation of type (p, ∞) . The relation between Jones polynomial (or the Kauffman bracket) and $col_3(L)$ has the form: $col_3(L) = 3|V(e^{\pi i/3})|^2$ and the formula relating the Kauffman polynomial and $col_5(L)$ has the form: $col_5(L) = 5|F(1, e^{2\pi i/5} + e^{-2\pi i/5})|^2$. This seems to suggest that the formula discovered by Jaeger involved Gaussian sums.

³³Our notation is based on Conway's notation for rational tangles. However, it differs from it by a sign change. The reason is that the Conway convention for a positive crossing is generally not used in the setting of skein relations.

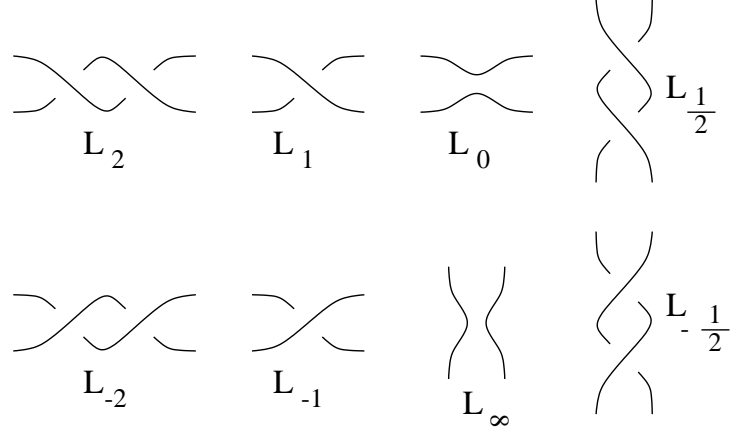


Fig. 3.5

If we rotate the figure from the relation (i) we obtain:

$$(i') \quad b_{-\frac{1}{2}}L_2 + b_{-1}L_1 + b_{\infty}L_0 + b_0L_{\infty} + b_1L_{-1} + b_2L_{-\frac{1}{2}} = 0$$

One can use (i) and (i') to eliminate $L_{-\frac{1}{2}}$ and to get the relation:

$$(b_2^2 - b_{-\frac{1}{2}}^2)L_2 + (b_1b_2 - b_{-1}b_{-\frac{1}{2}})L_1 + ((b_0b_2 - b_{\infty}b_{-\frac{1}{2}})L_0 + (b_{-1}b_2 - b_1b_{-\frac{1}{2}})L_{-1} + (b_{\infty}b_2 - b_0b_{-\frac{1}{2}})L_{\infty} = 0.$$

Thus, either we deal with the shorter relation (essentially the one in the fourth skein module described before) or all coefficients are equal to 0 and therefore (assuming that there are no zero divisors in R) $b_2 = \varepsilon b_{-\frac{1}{2}}$, $b_1 = \varepsilon b_{-1}$, and $b_0 = \varepsilon b_{\infty}$. Similarly, we would get: $b'_2 = \varepsilon b'_{-\frac{1}{2}}$, $b'_1 = \varepsilon b'_{-1}$, and $b'_0 = \varepsilon b'_{\infty}$, where $\varepsilon = \pm 1$. Assume, for simplicity, that $\varepsilon = 1$. Further relations among coefficients follow from the computation of the Hopf link component using the amphicheirality of the unoriented Hopf link. Namely, by comparing diagrams in Figure 3.6 and their mirror images we get

$$L \# H = -b_2^{-1}(b_1(a + a^{-1}) + a^{-2}b_2 + b_0(1 + T_1))L$$

$$L \# H = -b'_2{}^{-1}(b'_1(a + a^{-1}) + a^2b'_2 + b'_0(1 + T_1))L.$$

Possibly, the above equalities give the only other relations among coefficients (in the case of S^3). I would present below the simpler question (assuming $a = 1$, $b_x = b'_x$ and writing t^n for T_n).

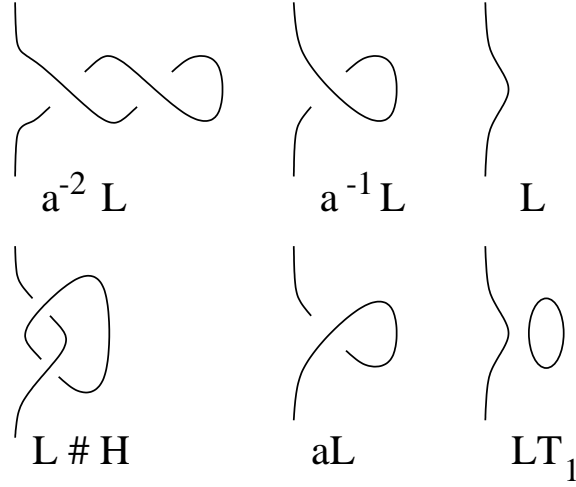


Fig. 3.6

Question 3.4 *Is there a polynomial invariant of unoriented links in S^3 , $P_{\frac{5}{2}}(L) \in \mathbb{Z}[b_0, b_1, t]$, which satisfies the following conditions?*

- (i) *Initial conditions: $P_{\frac{5}{2}}(T_n) = t^n$, where T_n is a trivial link of n components.*
- (ii) *Skein relations*

$$P_{\frac{5}{2}}(L_2) + b_1 P_{\frac{5}{2}}(L_1) + b_0 P_{\frac{5}{2}}(L_0) + b_0 P_{\frac{5}{2}}(L_\infty) + b_1 P_{\frac{5}{2}}(L_{-1}) + P_{\frac{5}{2}}(L_{-\frac{1}{2}}) = 0.$$

$$P_{\frac{5}{2}}(L_{-2}) + b_1 P_{\frac{5}{2}}(L_{-1}) + b_0 P_{\frac{5}{2}}(L_0) + b_0 P_{\frac{5}{2}}(L_\infty) + b_1 P_{\frac{5}{2}}(L_1) + P_{\frac{5}{2}}(L_{\frac{1}{2}}) = 0.$$

Notice that by taking the difference of our skein relations one gets the interesting identity:

$$P_{\frac{5}{2}}(L_2) - P_{\frac{5}{2}}(L_{-2}) = P_{\frac{5}{2}}(L_{\frac{1}{2}}) - P_{\frac{5}{2}}(L_{-\frac{1}{2}}).$$

Nobody has yet studied the skein module $\mathcal{S}_{\frac{5}{2}}(M; R)$ seriously so everything that you can find will be a new research, even a table of the polynomial $P_{\frac{5}{2}}(L)$ for small links, L .

I wish you luck!

4 Added in proof – the Montesinos-Nakanishi 3-move conjecture

A preliminary calculation performed by my student Mietek Dąbkowski (February 21, 2002) shows that the Montesinos-Nakanishi 3-move conjecture does not hold for the Chen link (Fig.1.14). Below is the text of the abstract we have sent for the Knots in Montreal conference organized by Steve Boyer and Adam Sikora in April 2002.

Authors: Mieczysław Dąbkowski, Józef H. Przytycki (GWU)

Title: Obstructions to the Montesinos-Nakanishi 3-move conjecture.

Yasutaka Nakanishi asked in 1981 whether a 3-move is an unknotting operation. This question is called, in the Kirby's problem list, *the Montesinos-Nakanishi Conjecture*. Various partial results have been obtained by Q.Chen, Y.Nakanishi, J.Przytycki and T.Tsukamoto. Nakanishi and Chen presented examples which they couldn't reduce (the Borromean rings and the closure of the square of the center of the fifth braid group, $\bar{\gamma}$, respectively). The only tool, to analyze 3-move equivalence, till 1999, was the Fox 3-coloring (the number of Fox 3-colorings is unchanged by a 3-move). It allowed to distinguish different trivial links but didn't separate Nakanishi and Chen examples from trivial links. The group of 3-colorings of a link L corresponds to the first homology group with Z_3 coefficients of the double branched cover of a link L , $M_L^{(2)}$, i.e.

$$Tri(L) = H_1(M_L^{(2)}, Z_3) \oplus Z_3$$

We find more delicate invariants of 3-moves using homotopy in place homology and we consider the fundamental group of $M_L^{(2)}$.

We define an n th Burnside group of a link as the quotient of the fundamental group of the double branched cover of the link divided by all relations of the form $a^n = 1$. For $n = 2, 3, 4, 6$ the quotient group is finite³⁴.

³⁴Burnside groups of links are instances of groups of finite exponents. Our method of analysis of tangle moves rely on the well developed theory of classical Burnside groups and the associated Lie rings. A group G is of a finite exponent if there is a finite integer n such that $g^n = e$ for all $g \in G$. If, in addition, there is no positive integer $m < n$ such that $g^m = e$ for all $g \in G$, then we say that G has an exponent n . Groups of finite exponents were considered for the first time by Burnside in 1902 [Bur]. In particular, Burnside himself was interested in the case when G is a finitely generated group of a fixed exponent. He asked the question, known as the Burnside Problem, whether there exist

The third Burnside group of a link is unchanged by 3-moves³⁵.

In the proof we use the "core" presentation of the group from the diagram; that is arcs are generators and each crossing gives a relation $c = ab^{-1}a$ where a corresponds to the overcrossing and b and c to undercrossings.

The Montesinos-Nakanishi 3-move conjecture does not hold for Chen's example $\hat{\gamma}$.

To show that $\hat{\gamma}$ has different third Burnside group than any trivial link it suffices to show that the following element, P , of the Burnside free group $B(4, 3) = \{x, y, z, t : (a)^3\}$ is nontrivial: $P = uwtu^{-1}w^{-1}t^{-1}$ where $u = xy^{-1}zt^{-1}$ and $w = x^{-1}yz^{-1}t$.

With the help of GAP it has been achieved!! (Feb. 21, 2002).

We have confirmed our calculation using also computer algebra system Magnus.

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To have a taste of Alexander letter, here is the quotation from the beginning of the interesting part: "When looking over Tait on knots among other things, He really doesn't get very far. He

infinite and finitely generated groups G of finite exponents.

Let $F_r = \langle x_1, x_2, \dots, x_r \mid - \rangle$ be the free group of rank r and let $B(r, n) = F_r/N$, where N is the normal subgroup of F_r generated by $\{g^n \mid g \in F_r\}$. The group $B(r, n)$ is known as the r th generator Burnside group of exponent n . In this notation, Burnside's question can be rephrased as follows. For what values of r and n is the Burnside group $B(r, n)$ finite? $B(1, n)$ is a cyclic group Z_n . $B(r, 2) = Z_2^n$. Burnside proved that $B(r, 3)$ is finite for all r and that $B(2, 4)$ is finite. In 1940 Sanov proved that $B(r, 4)$ is finite for all r , and in 1958 M.Hall proved that $B(r, 6)$ finite for all r . However, it was proved by Novikov and Adjan in 1968 that $B(r, n)$ is infinite whenever $r > 1$ and n is an odd and $n \geq 4381$ (this result was later improved by Adjan, who showed that $B(r, n)$ is infinite if $r > 1$ and n odd and $n \geq 665$). Sergei Ivanov proved that for $k \geq 48$ the group $B(2, 2^k)$ is infinite. Lysënok found that $B(2, 2^k)$ is infinite for $k \geq 13$. It is still an open problem though whether, for example, $B(2, 5)$, $B(2, 7)$ or $B(2, 8)$ are infinite or finite [VL, D-P-3].

³⁵ p th Burnside group is preserved by $\frac{ps}{q}$ -moves. This fact allows us to disprove Conjecture 2.2 [D-P-3].

merely writes down all the plane projections of knots with a limited number of crossings, tries out a few transformations that he happen to think of and assumes without proof that if he is unable to reduce one knot to another with a reasonable number of tries, the two are distinct. His invariant, the generalization of the Gaussian invariant ... for links is an invariant merely of the particular projection of the knot that you are dealing with, - the very thing I kept running up against in trying to get an integral that would apply. The same is true of his “Beknottednes”.

Here is a genuine and rather jolly invariant: take a plane projection of the knot and color alternate regions light blue (or if you prefer, baby pink). Walk all the way around the knot and ...”

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Department of Mathematics
George Washington University
Washington, DC 20052
USA
e-mail: przytyck@gwu.edu